Calculating Yaw of Repose and Spin Drift

-A novel and practical approach for computing the Spin Drift perturbation-

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Introduction

The Yaw of Repose angle β_R is a very small, but gradually increasing, horizontally rightward, aircraft-type yaw-attitude bias or "side-slip" angle of the ***coning axis*** of a right-hand spinning bullet. The Yaw of Repose reverses sign and angles leftward for a left-hand spinning bullet.

We discuss only right-hand spinning bullets here for clarity. It can be shown that for right-hand twist, the yaw of repose lies to the right of the trajectory. Thus the bullet cones around with an average attitude offset to the right, leading to increasing side drift to the right caused by a small rightward net aerodynamic lift-force. As we shall show, the yaw of repose is caused by the downward curving of the trajectory due to gravity. The yaw of repose is constrained to lie in a plane perpendicular to the gravity gradient.

For spin-stabilized bullets, this small rightward yaw attitude bias creates the well known rightward Spin Drift displacement. The small horizontally rightward yaw of repose angle causes a small rightward aerodynamic lift force which, in turn, causes a slowly increasing horizontal velocity of the bullet.

This effect occurs independently of the presence of surface wind of any force or from any direction. Bear in mind that the yaw of repose represents the horizontal yaw attitude of the bullet's coning axis or the average yaw of the coning bullet itself.

The acceleration of gravity acting upon the flat-fired bullet in free flight is the original cause of this small yaw-attitude bias angle. The downward curving of the trajectory due to gravity causes the airstream passing over the bullet to approach from below the nose of that bullet.

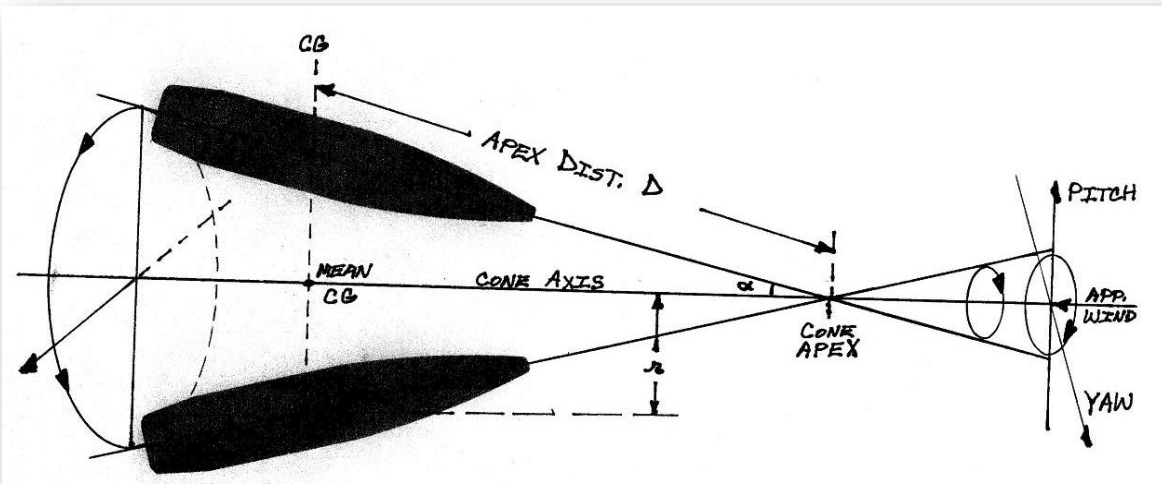


FIGURE: Extreme TDC and BDC Positions of Coning Bullet

This wind shift during each coning cycle causes an increased aerodynamic angle of attack which peaks as a *maximum* when the center of gravity (CG) of the bullet is at the Bottom Dead Center (BDC) position in each coning cycle where its nose is oriented maximally upward.

Since the coning angle always exceeds this small change in the approaching airstream direction during each coning cycle, the aerodynamic angle of attack for a bullet at Top Dead Center (TDC) when the bullet's nose is pointing maximally downward is at a *minimum* for that coning cycle, but that airstream continues to approach the bullet from below its axis of symmetry, even when the bullet is flying with its minimum possible coning angle.

The airflow approaching the coning bullet from beneath its coning axis does produce overturning torque vectors also lying in the horizontal plane when the coning bullet is located in the horizontal plane of the coning axis. These moments continually enlarge the coning motion to allow the coning axis to reorient itself into the new apparent wind direction. It is only the vertical-direction modulations of overturning moments which must be considered here, and they change sign and go through zero in the plane of the coning axis.

These modulations of aerodynamic angle of attack during each coning cycle produce small differential rightward-acting increments in the aerodynamic overturning moment experienced by the bullet which are centered upon the BDC or TDC positions of the bullet during each (lower or upper) half coning cycle.

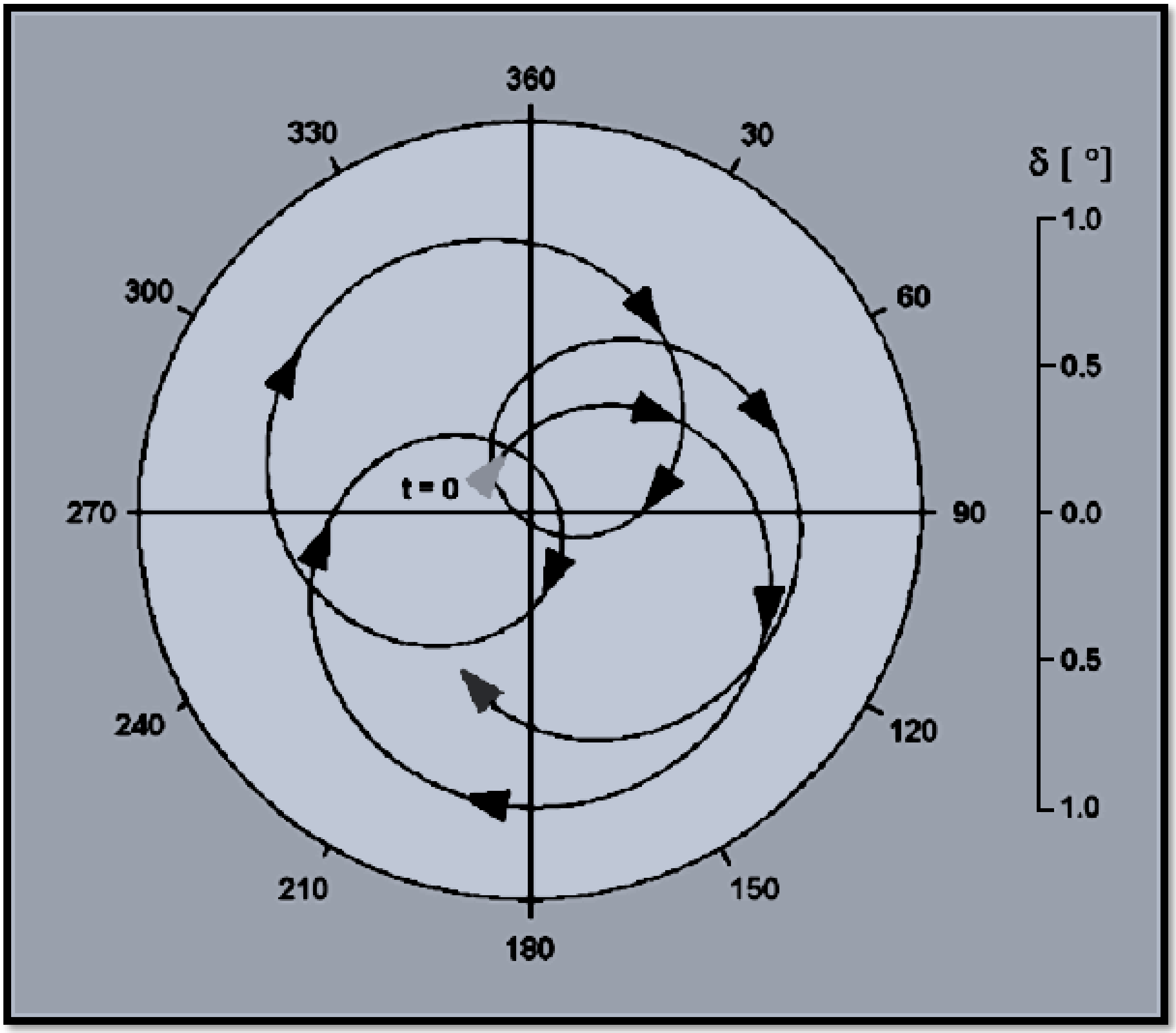
In physics these recurring differential torques are termed “torque impulses,” and each one pushes the forward-pointing angular momentum vector of the right-hand spinning bullet horizontally rightward without affecting its spin-rate or magnitude. Each torque impulse is evaluated by integration over the time of its half-coning cycle.

Both, the rotating overturning moment vector **M** due to the coning angle of attack **α** and its differential torque impulse vector **ΔM** inherently point *positive* rightward for the bullet at its BDC location. While the moment vector **M** itself points leftward at TDC, its *negative* differential torque vector **ΔM** is *positive rightward* as well at TDC. The differential torque is *negative leftward* at TDC because the aerodynamic angle of attack when the bullet is located there (or anywhere in the top half of the coning cycle) is *less* than the coning angle itself. The aerodynamic overturning moment is an *odd function* in the signed aerodynamic angle-of-attack, but we do not need to use that concept here.

Each rightward torque impulse **ΔM** tugs the forward-pointing angular momentum vector **L** slightly rightward along with the nose of the spinning bullet. The angular momentum vector **L** points *forward* along the spin-axis for a right-hand spinning bullet, as discussed here, and rearward for a left-hand spinning bullet. Each torque impulse **ΔM** is constrained to lie in the horizontal plane perpendicular to the gravity gradient because of the vector cross-product physical definition of torque.

The 175.16-grain M118LR 30-caliber bullet used as an example here is experiencing its 88th coning half-cycle when it reaches its target distance of 1,000 yards. The reinforcing cumulative effect of these rightward torque impulses occurring twice per coning cycle is the mechanism by which the downward arcing of the trajectory due to gravity causes the slowly increasing rightward yaw of repose attitude bias of the flying bullet.

The epicyclic motion of the spin-axis direction of a typical right-hand spinning rifle bullet is shown below for the first hundred yards, or so, of its flight. The gyroscopic stability **Sg** of this bullet at launch is about **1.33**


360
330
30
300
60
270
90
240
120
210
150
180
t = 0
δ [°]
1.0
0.5
0.0
0.5
1.0

General form of the yaw of repose, as described by BRL.

$$\delta_p = -\left(\frac{8I_x\varpi}{\rho\pi d^3 CM_a V_w^4}\right) V_w x \frac{dV_w}{d_t}$$

δ_p = Yaw of repose vector

CM_a = Aerodynamic Overturning Moment Coefficient

I_x = Axial Moment of Inertia

ϖ = Bullet Spin Rate

ρ = Air Density

d = Bullet Diameter

V_w = Velocity with respect to the air.

If the 3-dimensional mean trajectory of a rifle bullet in nearly horizontal flat firing is projected down onto a horizontal plane, the rightward deviation β_T of its tangent direction from the firing azimuth essentially defines this yaw of repose angle β_R throughout the flight, except for an even smaller horizontal dynamic tracking error angle ε_H as the trajectory curves to the right *following* (but lagging behind) the slightly larger yaw of repose angle β_R:

$$\beta_R = \beta_T + \varepsilon_H \qquad (1)$$

We will formulate a good approximation for β_T as an aid in formulating β_R accurately. The horizontal tracking error angle ε_H is *inherently non-negative* ($\varepsilon_H \geq 0$) for right-hand spinning bullets. The angle β_T also defines the horizontal-plane orientation of a mean CG-centered coordinate system moving with the **+V** direction of the flying bullet, with respect to the firing-point-centered earth-fixed coordinate system in which the trajectory is measured.

The yaw of repose has two effects on the trajectory of the projectile: 1) it produces a lateral lift-force that results in a the projectile drifting rightward (for a right-hand spinning projectile); and 2) it increases the total drag due by a small additional yaw-drag component.

The additional lift is of a very small magnitude, but cumulatively causes the rightward horizontal spin-drift displacement of the bullet's long-range trajectory. The accompanying additional yaw-drag component is an even smaller second-order term; thus, it is omitted. Any aerodynamic lift is always accompanied by some increase in aerodynamic drag.

Framework of the Analytical Solution

The horizontal spin-drift **SD** which we observe in long-range shooting is due to a horizontally acting aerodynamic *lift force* attributable to the increasing yaw of repose attitude angle β_R of the coning axis of the rightward-spinning rifle bullet.

We will use the principles of linear aeroballistics in formulating the yaw of repose β_R and its resulting spin-drift **SD**.

Detailed analyses of PRODAS 6-DoF simulation runs show that in flat firing the magnitude of the spin-drift **SD** in any given simulated firing is, beyond the first 150 yards or so, *very nearly equal* to some invariant scale factor **ScF** of about **1.0** to **2.4 percent**, more or less, times the bullet's *drop from the projected bore axis*:

$$SD(t) = -ScF*DROP(t) \qquad (2)$$

In other words, the horizontal spin-drift trajectory looks just like a small fraction **ScF** of the vertical trajectory rotated 90 degrees about the extended axis of the bore with each curvature ultimately caused by the same gravitational effect. The ratio of drift to drop rapidly approaches some particular **ScF** value for any rifle bullet asymptotically beyond the first 150 yards of that bullet's flight.

Our task is to formulate the scale factor **ScF** so that it can be accurately evaluated for any given bullet type and firing conditions.

Then using **Eq. 2**, we need only an accurate determination of the bullet's drop from the bore axis at the target distance to calculate an accurate spin-drift at any long-range target.

Existing 3-DoF "point mass" trajectory programs specialize in the accurate calculation of this bullet drop at the target distance in any firing conditions.

The Horizontal Tangent Angle

The instantaneous tangent to the horizontal-plane projection of the ***mean trajectory*** forms the angle $\beta_T(t)$ to an **X**-axis which defines the launch azimuth of the fired bullet in that horizontal plane. The mean trajectory of the bullet is the 3-dimensional path of the mean CG location, and which path would have been followed by the CG of the bullet if it were not coning about that mean trajectory. The mean trajectory is the path of the mean CG of the coning bullet.

This horizontal tangent angle $\beta_T(t)$ is always defined by the horizontal projection of the bullet's mean velocity vector $\mathbf{V(t)}$, but these mean velocity components are not calculated in our available PRODAS reports. The bullet's instantaneous cross-track velocity components are modulated by the helical coning motion of the CG of the bullet in flight.

Another important use for this horizontal tangent angle function $\beta_T(t)$ in ballistics is in plotting the horizontal yaw-attitude of the spin-axis of the bullet in the "wind axes" pitch-versus-yaw plots long used by ballisticians.

Just as the pitch coordinate data for the bullet's spin-axis direction is corrected by subtracting out the total change since launch $\Delta\Phi_{Total}(t) = \Phi(t) - \Phi(0)$ in the vertical-plane mean flight path angle Φ before plotting the pitch data for a 6-DoF simulated flight, the total change since launch in the mean trajectory's horizontal yaw-angle $\beta_T(t)$ should also be subtracted out before plotting of the bullet's yaw attitude data -- in the interest of logical consistency. However, this correction is not being done in contemporary aeroballistics. The resulting logical inconsistency stems from not fully understanding the coning motion of the flying bullet. The error persists because the angles involved are usually quite small.

With this change being made, the origin of the wind axes plots could truly be defined (horizontally as well as vertically) as the instantaneous $+\mathbf{V}$ direction of the bullet's ***mean trajectory***. Only the horizontal dynamic tracking error angle $\varepsilon_H(t)$ would remain in the plotted yaw-attitude values instead of the entire yaw-of-repose angle $\beta_R(t)$. A similar small *positive upward* vertical-direction dynamic tracking error angle $\varepsilon_V(t)$ *is* currently shown in these wind-axes plots of 6-DoF simulation results.

Because the scale factor **ScF** in **Eq. 2** is essentially invariant over time **t** and distance $\mathbf{x(t)}$ at long ranges, we can evaluate the trajectory's horizontal-plane tangent angle $\beta_T(t)$ directly from the bullet's vertical-plane **DROP** data in suitable distance units at any time **t** during its flight, by utilizing the small angle approximation that $\mathrm{Tan}(\alpha) \approx \alpha$ in radians:

$$\beta_T(t) = dSD/dx = -ScF^*[d(DROP)/dx] = -ScF^*[\Phi(t) - \Phi(0)] = -ScF^*\Delta\Phi(t) \qquad (3)$$

The **x**-derivative of **DROP(t)** can easily be seen to equal the Mean Flight Path Angle $\Phi(t)$ in horizontal firing when $\Phi(0)=0$, but it is clearly also $\Phi(t) - \Phi(0)$ when $\Phi(0)\neq 0$, whenever $\Phi(0)$ is small as in the flat-firing cases being considered here.

If the scale factor **ScF** is known, this expression allows calculation of the bullet's horizontal tangent angle in flat firing from either the ratio of its vertical drop rate to its forward velocity at any point during its flight or from the total change **ΔΦ(t)** in the vertical-plane mean flight path angle **Φ** since launch.

The "epicyclic swerve" modulation of the bullet's **DROP(t)** data rapidly fades to an insignificant fraction of the bullet's total **DROP** distance from the axis of the bore as the flight progresses.

The differential **DROP-rate** and remaining forward velocity **V(t)** are readily found from any simulated flight data. Calculation of the invariant scale factor **ScF** for any particular flight trajectory is discussed later in this paper.

Alternatively, one could evaluate $\beta_T(t)$ directly in the horizontal plane. As the bullet drifts horizontally due solely to spin-drift **SD(t)**, the intersection point with the **X**-axis of the tangent to the bullet's mean trajectory in the horizontal plane moves forward in the **+X** direction, but at a faster velocity than the forward velocity **V(t)** of the bullet itself. This intersection point starts about 150 yards later, but never quite catches up with the **X**-coordinate of the bullet in flat firing.

If we assume a continually increasing curvature of the horizontal trajectory so that this velocity ratio varies exponentially with range **X(t)**, we can estimate β_T, the dominant portion of β_R, as:

$$\beta_T \approx \mathrm{TAN}(\beta_T) = \mathrm{SD(t)}/\{\mathrm{X(t)}*0.825*\exp[-0.925*\mathrm{X(t)}/\mathrm{X(max)}]\} \qquad (4)$$

This hand-fitted estimator function agrees quite well with $\beta_T(t)$ angular values extracted from available trajectories generated by PRODAS 6-DoF simulations for the 1000-yard flight of our particular long-range rifle bullet by ratioing an extracted rightward $V_R(t)$ to **V(t)** for each millisecond of the PRODAS trajectory reports.

The horizontally rightward velocity component data $V_R(t)$ is extracted by applying a smoothing difference operator to the PRODAS "no wind, no Coriolis" drift data converted into linear units.

Comparing the two β_T functions for each millisecond over the **1.6923-second** simulated flight time yields a mean difference of **1.12 micro-radians** with a population standard deviation of **0.0514 milliradians**.

Extraction of the small rightward horizontal velocity V_R from the trajectory drift data is complicated by the superimposed epicyclic swerving of the CG of the bullet which accounts for most of the variance between these two functions.

The two approaches in **Eq. 3** and **Eq. 4** for evaluating $\beta_T(t)$ agree reasonably closely for PRODAS data as the epicyclic swerve modulations in **SD(t)** and **DROP(t)** dynamically damp out and fade into insignificance with ongoing flight time **t**.

We formulate these approximations for $\beta_T(t)$ so that they can be used as reasonableness checks on calculations of the yaw of repose $\beta_R(t)$ which is not itself reported by PRODAS.

We will eventually need an accurate formulation for $\beta_R(t)$ in order to calculate the scale factor **ScF** and thence the spin-drift **SD(t)** for other rifle bullet trajectories without relying upon 6-DoF simulation data.

PRODAS Simulated Flight Data

In this paper we will use as our example bullet the US Army's 30-caliber 175.16-grain "M118LR" bullet as was loaded in their M118LR Special Ball (7.62x51 mm NATO) long-range sniper and match ammunition in 2011.

We do this because we have several PRODAS 6-DoF simulation runs on hand from 2011 for this 7.62 mm NATO ammunition, reporting the linear ballistic results (including spin-drift) for each millisecond of its **1.6923-second** total simulated flight time to **1000 yards.**

The bullet weight actually used in these PRODAS runs is **175.16 grains.** The simulated firing conditions are 1) flat firing, 2) standard sea-level ICAO atmosphere, 3) no wind, 4) no Coliolis effect calculated, 5) muzzle velocity of **2600.07 feet per second**, and 6) barrel twist is right-handed at **11.5 inches per turn.**

The "no-wind" and "no-Coliolis" conditions assure that the rightward spin-drift **SD** is the only secular horizontal "bullet drift" being computed by PRODAS.

However, the PRODAS reported drift and drop data necessarily include the oscillating horizontal and vertical components of the bullet's helical coning motion about its mean trajectory throughout its simulated flight. We also have PRODAS runs available for this same bullet fired through constant left and right 10 MPH crosswinds as well as left-hand twist runs in each of the three constant wind conditions.

Yaw of Repose

We will show that in flat firing the continual downward arcing of the flight path angle **Φ** due to gravity causes repeated rightward differential aerodynamic torque impulses **ΔM** centered about the extreme top-dead-center (TDC) and bottom-dead-center (BDC) positions of the CG of the bullet in its coning motion.

These double-rate yaw attitude-changing horizontal torque impulses **ΔM** cause the forward-pointing angular momentum vector **L** of the right-hand spinning bullet to shift horizontally evermore rightward during its flight. In light of Coning Theory, we should more precisely say the bullet's ***coning axis*** drifts horizontally rightward in its yaw attitude throughout the flight.

Ballisticians term this accumulating yaw-attitude bias the "yaw of repose" angle β_R of the flying bullet and classically formulate its horizontal component from calculus as [**Eq.10.83** in McCoy's MEB]:

$$\beta_R = P*G/M \tag{5}$$

This expression is the horizontal part of the *particular solution* for the differential Equations of Motion which determine each trajectory in terms of the classic aeroballistic auxiliary parameters:

$$P = (Ix/Iy)*p*d/V = (\omega_1 + \omega_2)*d/V \tag{6}$$

$$G = g*d*Cos(\Phi)/V^2 \approx g*d/V^2 \tag{7}$$

$$M = (m*d^2/Iy)*[\rho*S*d/(2*m)]*CM\alpha = (\omega_1 + \omega_2)*\omega_2*d^2/V^2 \tag{8}$$

after converting each classic auxiliary parameter from dimensionless (canonical) arc-length-rates into the time-rate units used in our physical analyses of flat-firing a spin-stabilized rifle bullet.

The change-of-variables in **Eq. 6** uses one of the gyroscopic relationships from Tri-Cyclic Theory [Harold Vaughn of Sandia Labs and Dr. John D. Nicolaides, 1953] that:

$$(Ix/Iy)*p = (Ix/Iy)*\omega = \omega_1 + \omega_2 = \omega_2*(R+1) \tag{9}$$

where $R = \omega_1/\omega_2$ is called the "stability ratio" which is perfectly 1:1 corelated with the gyroscopic stability $Sg = (R+1)^2/(4*R)$.

McCoy defines the spin-rate **p** of the bullet as used here to be a circular frequency given in radians per second. The bullet's spin-rate **p** is sometimes given elsewhere in aeroballistics in units of revolutions per second (or hertz), and is sometimes given in radians per foot of bullet travel, or even in radians per caliber **d** of bullet travel.

To avoid this confusion we use the more conventional symbol **ω** here for the circular frequency of the spin-rate of the bullet given in **radians per second**. We also use the symbol **f** for rotation rates as frequencies in revolutions per second or **hertz.**

The change of variables in **Eq. 8** uses the fundamental magnitude relationship from Coning Theory that:

$$(\rho*S*V^2/2)*d*CM\alpha = L*\omega_2 = (Ix*\omega)*\omega_2 \qquad (10)$$

as well as the Tri-Cyclic relation in **Eq. 9** again. Note that this Coning Theory relationship implies that the slow-mode coning rate ω_2 in radians per second is always given by **(M/P)*V/d** in terms of the canonical aeroballistic auxiliary parameters **M** and **P**.

With these changes of variables, the classic formulation for the yaw of repose angle β_R reduces to:

$$\beta_R = g/[\omega_2(t)*V(t)] = g/[2\pi*f_2(t)*V(t)] \qquad (11)$$

While this formulation for β_R is classic, it *does not inherently yield* **zero** *at* **t = 0**, and it is about a factor of **π** too small at long ranges when compared to β_T as formulated above [**Eq. 1** and **Eq. 3**].

Let us say the mean flight path angle **Φ** of the bullet's trajectory changes downward by a small decrement **ΔΦ** due solely to the pull of gravity (as with a vacuum trajectory) during one-half of the period T_2 of a particular coning cycle. As a continuous variable in flight time **t**, this angular decrement **ΔΦ(t) = 0.0** at **t = 0** by definition.

In flat firing, the small decrement **ΔΦ** in the nearly horizontal flight path angle **Φ(t)** during the time interval $T_2/2$ of a particular half-coning cycle can be expressed as:

$$\Delta\Phi(t) \approx TAN(\Delta\Phi) = -(g*T_2)/[2*V(t)] = -g/[2*f_2(t)*V(t)] = -\pi*g/[\omega_2(t)*V(t)] \qquad (12)$$

where $f_2(t)$ is the instantaneous coning rate, or gyroscopic precession rate, of the bullet given in revolutions per second, or hertz. [Here we are ignoring the significant cross-bore-axis (upward) component of the real bullet's ballistic drag force F_D in interest of formulating a simple **SD** estimator. This oversimplification will be explained and dealt with later.]

Comparing our version of the classic formulation for the steady-state yaw of repose $\beta_R(t)$ in **Eq. 11** with the change in flight path angle **ΔΦ(t)** due solely to gravity *per half-coning cycle* above, we note that:

$$\beta_R(t) = (-1/\pi)*\Delta\Phi(t) \qquad (13)$$

Thus, our formulation in **Eq. 12** above for **ΔΦ(t)**, the change in flight path angle **Φ** per half-coning cycle $T_2/2$ which *does* inherently equal **zero** at **t = 0**, actually looks like a more suitable formulation for $\beta_R(t)$ than does the classic form.

We will now investigate the aerodynamic and gyroscopic causes of $\beta_R(t)$ so that we can formulate its value at any time **t** during the flight of any rifle bullet.

At each extreme vertical location, TDC and BDC, the coning bullet experiences a peak rate of differential change in its aerodynamic overturning moment vector **M** due to this differential change **ΔΦ** in its vertical-direction (upward airflow) aerodynamic angle-of-attack. Each of these two differential torque impulse vectors **ΔM** points *horizontally rightward* as seen from behind the right-hand spinning bullet.

Here these two differential torque *impulse* vectors **ΔM** are to be evaluated by integrating the differential torque over each half (upper or lower) of the coning period T_2, giving them units of torque multiplied by time which correspond with the units of angular momentum.

Owing to the increased aerodynamic angle-of-attack of the apparent wind experienced by the bullet at its BDC position, the differential torque impulse **ΔM** at BDC is *inherently positive rightward*, temporarily increasing the overturning moment **M** acting upon the bullet at this BDC location in its coning motion.

While the overturning moment vector **M** itself points *leftward* at the TDC position of the bullet, the differential torque impulse vector **ΔM** is *inherently negative* due to the *reduced* aerodynamic angle-of-attack experienced by the coning bullet at that upper location, and so the differential torque impulse vector **ΔM** itself points *positive rightward*, once again, at TDC.

Thus, the alternating sequences of TDC and BDC differential torque impulses are *mutually reinforcing* throughout the bullet's flight.

Recall that in Coning Theory the spin-axis of the bullet is pointing maximally *upward* when the CG of the bullet is at its BDC position in any coning cycle; i.e., its aerodynamic pitch attitude is a relative maximum during that coning cycle.

As formulated in linear aeroballistics, the instantaneous magnitude **{M}** of the overturning moment **M** at time **t** is:

$$\{M\} = q*S*d*\mathrm{Sin}[\alpha(t)]*CM\alpha$$

Where

$q = (\rho/2)*V^2 =$ Dynamic Pressure in lbf/square foot.

$\rho =$ Density of the atmosphere **= 0.0764742 lbm/cubic foot** for the standard sea-level ICAO atmosphere used here. This value of the density **ρ** must be divided by the acceleration of gravity **g = 32.174 feet per second per second** to convert its units into proper density units, mass (**slugs**) per cubic foot.

V = Airspeed of the bullet in feet/second.

S = Reference (frontal) area of the bullet at the base of its ogive in square feet $= (\pi/4)*d^2$.

d = Diameter of the bullet in feet.

$\alpha(t)$ = Coning angle (and aerodynamic angle-of-attack) of the bullet in radians at any time **t** during its flight.

$CM\alpha$ = Dimensionless overturning moment coefficient in linear aeroballistics theory.

Here we are ignoring the fast-mode gyroscopic nutation $\omega_1(t)$ of the bullet's spin-axis for several reasons: 1) It does not normally move the CG of the bullet by any measurable amount, 2) Its aeroballistic effects tend to average out to zero rather rapidly, and 3) It rapidly damps to insignificance for most rifle bullets after any flight disturbance.

As the flat-firing trajectory of the coning bullet, flying essentially horizontally near the **X**-axis (with $\Phi \approx 0.0$ and with its coning axis aligned into the approaching windstream), arcs downward due to gravity, the aerodynamic angle-of-attack $\alpha(t)$ increases by the magnitude of $\Delta\Phi$ at its BDC location in this coning cycle.

The **cosines** of small coning angles $\alpha(t)<5.7$ **degrees**, the flight path angle Φ, and the small change in flight path angle $\Delta\Phi$ all remain essentially equal **1.00**. From trigonometry, the peak magnitude $\{\Delta M\}_{PEAK}$ of this differential overturning torque ΔM with the bullet at its BDC location can be expressed as:

$$Sin(\alpha + \Delta\Phi) = Sin(\alpha)*Cos(\Delta\Phi) + Cos(\alpha)*Sin(\Delta\Phi) \approx Sin(\alpha) + Sin(\Delta\Phi)$$

$$M + \{\Delta M\}_{PEAK} = q*S*d*Sin(\alpha + \Delta\Phi)*CM\alpha \approx M + q*S*d*Sin(\Delta\Phi)*CM\alpha$$

$$\{\Delta M\}_{PEAK} = q*S*d*Sin(\Delta\Phi)*CM\alpha \qquad (14)$$

This expression can also be well approximated as:

$$\{\Delta M\}_{PEAK} = q*S*d*(\Delta\Phi)*CM\alpha \qquad (15)$$

The instantaneous vertical-direction aerodynamic angle-of-attack is actually the vector sum of three small angles in complex wind-axes coordinates (ignoring the fast-mode ω_1 motion):

1. Vertical component of the slow-mode coning angle, $\alpha(t)*Cos(\omega_2*t + \xi_0)$
2. Downward change in flight path angle $\Delta\Phi$, and
3. Very small vertical-direction tracking error angle ε_V (upward in wind-axes plots). This vertical-direction tracking error angle ε_V is termed the "pitch-of-repose" by McCoy.

The primary overturning moment **M** is due to (1.) the coning angle-of-attack $\alpha(t)$. This rotating torque vector **M** produces the slow-mode circular coning motion of the CG of the bullet at the coning rate $\omega_2(t)$ of the bullet as a gyroscopic precession of the bullet's spin-axis.

Examination of several different PRODAS runs shows that even for a *dynamically stable* bullet with any early coning motion fully damped down, **$\alpha(t)$** always exceeds **$\Delta\Phi$** by some margin all the way to maximum supersonic range and beyond.

From Coning Theory, the vertical component of the complex coning angle **$\alpha(t)$** is the "pitch angle" **$\varphi(t)$** given by the *real part* of the complex **$\alpha(t)$**, again neglecting the fast-mode motion:

$$\varphi(t) = Re[\alpha(t)] = \{\alpha(t)\}*Cos(\omega_2*t + \xi_0) \quad (16)$$

Whenever **$\{\alpha(t)\}>>\Delta\Phi$**, only the portion of **$\varphi(t)$** equal in magnitude to **$\Delta\Phi$** produces the differential overturning moment impulse **ΔM** which drives the spin-axis of the bullet rightward giving rise to the bullet's Yaw of Repose angle **β_R**, and the overturning moment impulses at BDC and TDC can be modeled as having the same form.

The excess of **$\varphi(t)$** over **$\Delta\Phi$** goes toward enlarging the coning angle **$\alpha(t)$**, counteracting any frictional aerodynamic damping of that slow-mode coning motion of the bullet.

The instantaneous differential overturning moment {**ΔM**} is then due to the vertical-direction differential aerodynamic angle-of-attack **$\Delta\Phi(t)*Cos(\omega_2*t + \xi_0)$**.

This modulation at the coning-rate **ω_2** looks like a full-wave-rectified sine wave over each coning cycle. The time-average over each quarter wave is just **$2/\pi$** times the peak value.

The average value of **$\Delta\Phi$** itself over each half-coning cycle is just **$\Delta\Phi/2$** because the flight path angle **Φ** varies almost linearly over the small interval **$T_2/2$**. Averaged over the top or bottom one-half of a coning cycle, the average effective angle-of-attack is then **$(2/\pi)*\Delta\Phi/2 = \Delta\Phi/\pi$**.

The vector sum of (2) **$\Delta\Phi$** and (3) **ε_V** varies only gradually with ongoing time-of-flight **t**. The *magnitudes* of these two small angles *sum* to an average vertical-direction aerodynamic angle-of-attack which drives the coning-axis direction *continually downward* according to Coning Theory, dynamically tracking (but lagging behind) the downward-curving trajectory.

The time-integrated ***torque impulse*** **ΔM** centered at TDC or BDC must equal the differential torque due to the time-average **$\Delta\Phi/\pi$** of the modulated aerodynamic angle-of-attack multiplied by the total time interval **$T_2/2$** for each half-coning cycle. The interval **T_2** increases gradually as the coning rate **$\omega_2(t)$** slows throughout the flight.

The effective differential torque impulse **ΔM** integrated over a particular *half-coning cycle* thus becomes:

$$\Delta M = (T_2/2)*q*S*d*(\Delta\Phi/\pi)*CM\alpha \quad (17)$$

Substituting the unsigned *magnitude* of the first expression for **$\Delta\Phi$** from **Eq. 12** yields:

$$\Delta M = (1/\pi)*g*(T_2/2)^2 *q*S*d*CM\alpha/V(t) \quad (18)$$

This differential torque impulse **ΔM** has units of lbf-feet-seconds which can be converted into slug-feet squared per second, a proper set of units for angular momentum.

The right-hand spinning bullet alters its pointing direction *rightward* in gyroscopic reaction to each of these two differential torque impulses **ΔM** during each coning cycle. However, it does so in an unusual way.

When a constant-magnitude, rotating overturning moment vector **M** is applied to a spinning gyroscope, its spin-axis direction soon begins moving in precession and nutation in reaction to that steadily rotating torque vector.

However, the first motion of its spin-axis is always in the direction of the eccentric force producing the overturning moment **M** while those epicyclic motions are getting started.

For the spinning bullet, the eccentric force is the total aerodynamic force **F**, a line vector acting through the aerodynamic center-of-pressure CP of the bullet at any instant during its flight. For spin-stabilized, rotationally symmetric rifle bullets, the CP is nearly always located forward of the CG along the spin-axis of the bullet.

In response to each small torque *impulse* **ΔM**, the spin-axis of our bullet moves *initially rightward*, but each impulse ceases well before any vertically upward or downward movement of the spin-axis can become established.

When the torque impulse vector **ΔM** is expressed in the same units as the angular momentum vector **L** of the spinning bullet, having physical dimensions of mass times length squared over time, their *direct vector sum* defines the resulting angular momentum **L** of the spinning bullet after the torque impulse has been applied.

For a right-hand spinning bullet the angular momentum vector **L** points forward along its spin-axis. Here, since **ΔM** is always acting perpendicularly to **L**, the *direction* of the angular momentum vector **L** is shifted rightward by an incremental angular amount (in radians), which we term $\Delta\beta_R$, but its *magnitude* remains unchanged.

Of course, the nose of the right-hand spinning rifle bullet in stable supersonic flight always points in the direction of its angular momentum vector **L**.

The incremental increase $\Delta\beta_R$ in the yaw of repose angle β_R during each *half coning cycle* is thus:

$$\Delta\beta_R \approx \mathrm{Tan}(\Delta\beta_R) = \{\Delta M\}/\{L\} = (1/\pi)*g*(T_2/2)^2*q*S*d*CM\alpha/[L*V(t)] \qquad (19)$$

Recalling **Eq. 22** from the Coning Theory paper, we note that the right-hand side of **Eq. 19** above contains the fundamental expression from Coning Theory for determining the *magnitude* of the circular coning rate $\omega_2(t)$ for a spin-stabilized bullet coning at non-zero angles of attack α:

$\omega_2 = q*S*d*CM\alpha/L \qquad\qquad (\alpha, L \neq 0)$

Due to the acceleration of gravity, the coning angle **α(t)** cannot be **zero** in flat firing except perhaps very briefly at **t = 0**, where this magnitude relationship still holds true.

After this change of variables,

$$\Delta\beta_R = (1/\pi)*g*(T_2^2)*\omega_2(t)/[4*V(t)] \qquad (20)$$

This change of variables is critically important in formulating an analytical calculation of β_R because it simultaneously eliminates from the formulation both the overturning moment coefficient **CMα** and the angular momentum **L** of the bullet, each of which is difficult to calculate for a new bullet. The coning rate $\omega_2(t)$ is more readily obtainable from Tri-Cyclic Theory.

Also recall that by definition $T_2^2 = 1/(f_2)^2 = 4\pi^2/\omega_2^2$. After this substitution we have:

$$\Delta\beta_R = \pi*g*/[\omega_2(t)*V(t)] = g/[2*f_2(t)*V(t)] = g*T_2/[2*V(t)] = -\Delta\Phi \qquad (21)$$

While this expression is dimensionless, the increment in the yaw of repose angle $\Delta\beta_R$ for each half-coning-cycle $T_2/2$ is calculated here in **radians.** The proper algebraic sign depends upon coordinate system conventions and the sense of the bullet's spin-rate.

In linear aeroballistics theory, the instantaneous aerodynamic lift-force driving the spin-drift **SD** of the bullet's CG horizontally rightward from the **X**-axis is *linearly* proportional to the aerodynamic angle-of-attack for the very small yaw of repose angle β_R.

Thus, the linear dependence of $\Delta\beta_R$ upon **ΔΦ** shown in **Eq. 21** explains the remarkable similarity in shape of the horizontal-plane and vertical-plane projections of the bullet's "no wind, no Coriolis" mean trajectory in 3-space.

We could have arrived at the result shown in **Eq. 21** more directly by assuming unrealistically that the flying bullet was not coning, but simply spinning about its coning axis direction with a **zero** coning angle **α**, or by assuming that a **non-zero** coning angle **α** does not matter. But then we would have to validate either of those assumptions as we have done above.

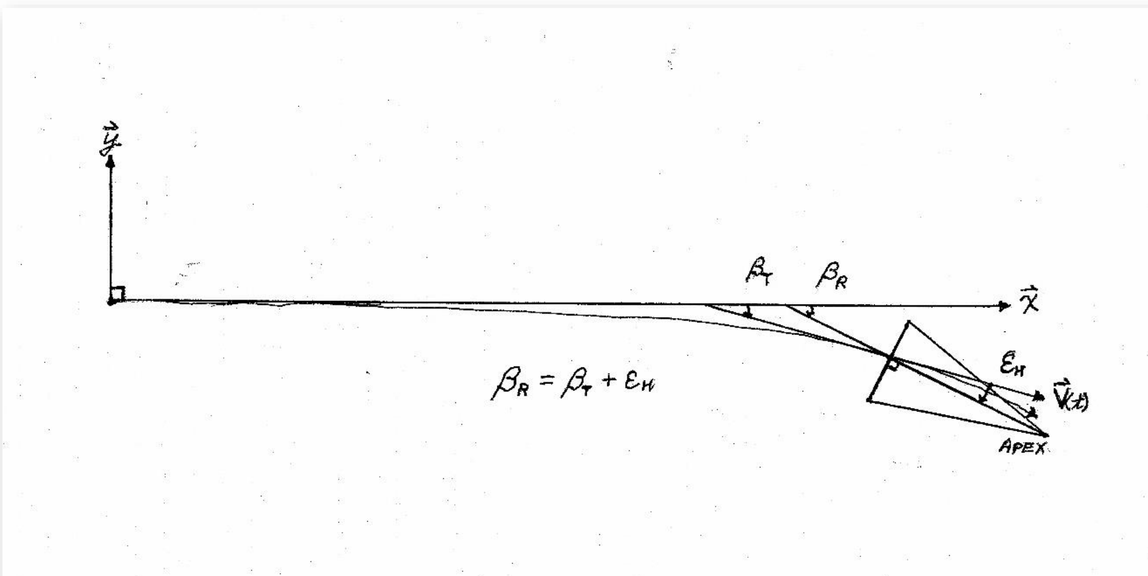
For minimum coning angle flight, when $\alpha(t) \gtrsim \delta \approx \Delta\Phi$ as becomes the case eventually in most "constant wind" 6-DoF simulations, the average torque impulses **ΔM** are no longer precisely symmetrically equal at BDC and TDC. In fact, **ΔM(BDC)≳ΔM(TDC)**, and their combined average effect would be slightly smaller (by about 5 percent) than these estimates here yielding a maximal yaw of repose angle.

The yaw of repose angle $\beta_R(t)$ can be found by *summing* the increments $\Delta\beta_R$ divided by $T_2/2$ for each *half coning cycle* which has occurred from **t = 0** to time **t**, starting with $\beta_R(0)$ equal **zero**.

Using the data from "no wind, no Coriolis" PRODAS reports for this 30-caliber bullet, yields **β_R(1.430 sec) = 0.67208 milliradians**, which exceeds our fitted value of **β_T(1.430 sec) = 0.61019 mrad** by **10.14 percent**.

We term this difference, **β_R(t) – β_T(t),** the ***horizontal tracking error angle* ε_H(t)**. We are comparing these angles here at **t = 1.430 seconds** after launch when this M118LR bullet has slowed to Mach 1.20 or **1340 feet per second** at **888.5 yards** downrange in these simulated firing conditions.

Using the PRODAS-calculated velocity and coning-rate data, our adjusted version of the classic formulation of the yaw of repose yields **β_R(1.430 sec) = π*P*G/M = 0.70633 milliradians**, which exceeds our fitted value of **β_T(1.430 sec) = 0.61019 mrad** by **15.76 percent** for the horizontal dynamic tracking error angle **ε_H.** We believe this adjusted classic formulation for **β_R** better matches the case for a significantly coning bullet than for this particular minimally coning PRODAS trajectory.



Projection of the Mean Trajectory onto the Horizontal Plane.

Estimating the Yaw of Repose

In the absence of having 6-DoF simulation data available, we could approximate the yaw of repose angle **$\beta_R(t)$** by assigning readily integrable (in closed form) continuous functions of time **t** to represent the variables **$\omega_2(t)$** and **V(t)** in **Eq. 21** so that we could then approximate this *summing* operation by performing the definite integration of **$\Delta\beta_R(t)$** over time from **t = 0** to time **t** and dividing the integrated result by the total time interval **t**:

$$\beta_R(t) = (2\pi^*g/t)\int[\omega_2(t)^*V(t)]^{-1}\,dt \qquad (22)$$

Here the extra factor of **2** in this expression for **$\beta_R(t)$** in **Eq. 22** versus the expression for **$\Delta\beta_R(t)$** in **Eq. 21** is due to integrating **$\Delta\Phi(t)$** continuously rather than using its *average value* **$\Delta\Phi/2$** over each half-coning cycle.

Note that the size of the yaw of repose angle **$\beta_R(t)$** whenever **$\alpha(t) >> \Delta\Phi$** depends only on the velocity **V(t)** and coning rate **$\omega_2(t)$** of the bullet as functions of time. In particular, **$\beta_R(t)$** in this formulation is independent of the coning angle **$\alpha(t)$** itself in this analysis**.**

Since the spin-drift displacement **SD(t)** is caused directly by this yaw of repose angle **$\beta_R(t)$** as an aerodynamic lift effect, evaluation of the spin-drift **SD(t)** does not require detailed knowledge of the bullet's coning angle **$\alpha(t)$**. This independence of **$\beta_R(t)$** is significant because the coning angle **$\alpha(t)$** is a *free variable* in Coning Theory and is thus difficult to evaluate analytically except in special cases.

If we find the values of **$\omega_2(t)$** and **V(t)** at **t = 0** and at a much later flight time **t = T**, and we assume for approximation purposes that each function decays exponentially with time **t**, then the definite integral for **$\beta_R(t)$** can be expressed as:

$$\beta_R(t) = \{2\pi^*g/[\omega_2(0)^*V(0)^*T]\}\int\exp[-(k_\omega + k_V)^*t/T]\,dt \qquad (23)$$

with

$$k_\omega = \ln[(\omega_2(T)/\omega_2(0)]$$

$$\omega_2(t) = \omega_2(0)^*\exp[k_\omega{}^*t/T] \qquad (24)$$

The decay in bullet spin-rate **$d\omega/dt$** is due to skin friction, a torque **Γ_S** about the **x**-axis of the bullet, which opposes and slows its spin-rate **ω**:

$$\Gamma_S = dL/dt = d/dt[I_x{}^*\omega] = I_x{}^*d\omega/dt.$$

This skin friction torque **Γ_S** itself is proportional to the circumference of the bullet **π^*d**.

The second moment of inertia of the spinning bullet about its spin-axis is given by

$$I_x = m^*d^2\,{}^*k_x{}^2$$

So, the time rate of change in spin-rate **$d\omega/dt$** is inversely proportional to the caliber **d^{-1}** of the bullet.

The decay in the spin-rate **$\omega(t)$** for any modern rifle bullet in flight is closely approximated by the exponential expression:

$\omega(t) \approx \omega(0)*\exp[-(0.0321/d)*t]$ (with d = caliber in inches)

The indicated spin decay-rate coefficient **-0.0321/d** closely matches the spin-rates shown throughout the PRODAS runs for the M118LR bullet having **0.308-inch** diameter **d**.

From the Tri-Cyclic Theory, we can evaluate the coning rate **$\omega_2(t)$** as

$$\omega_2(t) = (Ix/Iy)*\omega(t)/[R(t) + 1]$$

$$\omega_2(T)/\omega_2(0) = [\omega(T)/\omega(0)]*\{[R(0) + 1]/[R(T) + 1]\}$$

$$k\omega = \ln[\omega_2(T)/\omega_2(0)] = [-(0.0321/d)*T] + \ln\{[R(0) + 1]/[R(T) + 1]\}$$

$$k\omega \approx [-(0.0321/d)*T] + [-0.585*T] = -(0.585 + 0.0321/d)*T$$

This approximation provides a better than 5-percent fit to the coning rates **$\omega_2(t)$** calculated in PRODAS for this M118LR example bullet for each millisecond of its flight to 1000 yards, so we shall use this approximation as well for other rifle bullets pending further analysis.

We also approximate the decay-rate in bullet velocity **V(t)** as exponential in time **t**:

$$k_V = \ln[V(T)/V(0)]$$

$$V(t) \approx V(0)*\exp[k_V*t/T] \qquad (25)$$

Here we are using **t/T** as a dimensionless canonical variable in the exponential decay expressions and as a dummy variable in the (summing) integration.

After the definite integration from **t = 0** to **t =T**, the expression for **$\beta_R(t)$** is:

$$\beta_R(t) = \{-2\pi*g/[\omega_2(0)*V(0)*(k\omega + k_V)]\}*\{\exp[-(k\omega + k_V)*t/T] - 1\} \qquad (26)$$

Note that **$\beta_R(0) = 0.00$** as we require here.

Our "no wind anywhere" PRODAS runs for this M118LR bullet show that **V(t)** slows from an initial velocity of **2600.07 feet per second** to **1340 FPS** (Mach 1.20) at **888.5 yards** downrange with a time-of-flight (**T**) of **1.430 seconds**, and that the coning rate **$\omega_2(t)$** of the bullet slows from **2π*45.57 radians per second** to **2π*17.00 radians per second** over this same interval **T**.

The yaw of repose angle **$\beta_R(T)$** at **T = 1.430 seconds**, and at **888.5 yards** downrange, would then be calculated as:

$$k\omega \approx -(0.585 + 0.0321/d)*T = -0.98559$$

$k_V = \ln[V(T)/V(0)] = -0.66328$

$\beta_R(1.430\ sec) = [1.6469\text{x}10^{-4}]*[4.2011] = 0.69186\ mrad = 0.039641\ degrees$ (27)

While PRODAS does not report β_R, this small **0.040-degree** angle is not unreasonable for this bullet at **888.5 yards** downrange.

A smoothed value of **0.5040 milliradians** (or **0.02888 degrees**) can be directly calculated for the *tangent angle* β_T at 888.5 yards into a "no wind" PRODAS simulated flight by ratioing an extracted horizontally rightward velocity $V_R(t)$ to the forward velocity $V(t)$ of the bullet at **t=1.430 seconds**.

However, this velocity ratio is very sensitive to the ongoing epicyclic swerving motion included in the PRODAS Drift reports, and its smoothed value probably should be somewhat larger here at **t = 1.430 seconds**.

Our fitted algorithm, mentioned above, for estimating the tangent angle β_T at 888.5 yards yields **0.61019 milliradians** or **0.034961 degrees**. This would indicate a reasonable horizontal tracking error angle ε_H of **0.08167 milliradians**, or **13.38 percent** of β_T at that point in the flight.

This closed-form integration yields a value of β_R about midway between our numerically-integrated value and the adjusted classic value of β_R as calculated from the same PRODAS data at **t = T**.

We shall use this closed-form algorithm (**Eq. 26**) for estimating the yaw of repose angle $\beta_R(t)$ without relying upon any 6-DoF simulation data in formulating the spin-drift **SD(t)** of any rifle bullet at long ranges.

Think of these incremental yaw-attitude changes as occurring twice per coning cycle at the TDC and BDC positions of the coning bullet throughout the flight.

The double-coning-rate sequence of small torque impulses **ΔM** produces a reinforcing chain of these "first motions" which gradually shifts the coning-axis direction of the spinning bullet evermore rightward.

The initial yaw of repose angle at bullet launch $\beta_R(0)$ must be **zero** by definition.

These calculations serve to validate our analysis of the gyroscopic and aerodynamic causes of the yaw of repose.

Analysis of the Spin Drift

The horizontally rightward spin-drift **SD(t)** of the trajectory is caused by a net horizontal aerodynamic lift-force attributable to this small, but ever increasing, rightward yaw of repose angular bias $\beta_R(t)$ in the yaw-attitude of the coning-axis of the spinning bullet.

The pointing direction of the bullet's coning axis quickly tracks each of these small changes in the approaching apparent wind direction within one half of a coning cycle, just as with any other type of wind change.

As the horizontal projection of the ***mean trajectory*** traced by the ***mean CG*** of the bullet gradually accelerates rightward with this spin-drift **SD(t)**, its tangent **+V** direction defining the origin of wind-axes plots drifts slowly rightward also, *following* (but dynamically lagging) the increasing yaw of repose attitude angle $\beta_R(t)$ of the bullet.

We formulated this tangent angle $\beta_T(t)$ earlier. Logically, only the horizontal tracking error angle $\varepsilon_H(t) = \beta_R(t) - \beta_T(t) \geq 0$ should appear in these wind-axes plots in place of $\beta_R(t)$, itself.

In formulating the effective net (time-averaged) aerodynamic lift-force accelerating the CG of the coning bullet rightward, we must consider the coning modulation of the aerodynamic effect as the CG of the bullet moves throughout its circular coning cycle.

Here the modulation is horizontally left-to-right, and the effect being modulated is an aerodynamic lift force.

However, as we saw above for the modulation of the overturning moment, for the uniformly coning rifle bullet, analysis of the modulation of this lift force can be greatly simplified by making use of Coning Theory. We can express the average effective aerodynamic lift-force on the coning bullet arising from the yaw of repose angle $\beta_R(t)$ as if the bullet were *not* coning, but simply flying with the spin-axis always aligned with the attitude of its coning axis $[\alpha(t) = 0]$. After all, it is the attitude of that coning axis which properly defines this yaw of repose angle $\beta_R(t)$.

The actual average aerodynamic angle of attack in a coordinate system moving with, and oriented with the mean trajectory of the coning bullet, is just the tracking error angle $\varepsilon_H(t)$. The lift-force attributable to this $\varepsilon_H(t)$ angle of attack keeps increasing the rightward curvature of the mean trajectory. In earth-fixed coordinates, not oriented to $\beta_T(t)$ with the yawing bullet, the average horizontal angle of attack driving the mean trajectory away from the original firing azimuth, the **X**-axis, is $\varepsilon_H(t)+\beta_T(t) = \beta_R(t)$.

The magnitude of the small net rightward aerodynamic lift force $\{F_L\}_R$ attributable to the rightward yaw attitude bias $\beta_R(t)$ of the coning axis is given in linear aeroballistics as:

$$\{F_L\}_R = q*S*\{CL_\beta*Sin[\beta_T(t) + \varepsilon_H(t)] - CD*Sin[\beta_T(t)]\}$$

Or

$\{F_L\}_R \approx q(t)*S*CL_\beta(t)*\beta_R(t) - q(t)*S*CD(t)*\beta_T(t)$ (29)

The small rightward aerodynamic lift force acting horizontally on the bullet is actually counteracted partially by an even smaller cross-bore component of the bullet's significant aerodynamic drag force F_D given by $q*S*CD*\beta_T(t)$.

Here the coefficient of lift $CL_\beta(t)$ and coefficient of drag $CD(t)$ are evaluated for the very small aerodynamic angle-of-attack $\beta_R(t)$. However, they still vary with the Mach-speed of the slowing bullet. The dynamic pressure **q(t)** also reduces with the square of its airspeed **V(t)** as the bullet slows.

This small rightward horizontal force $\{F_L\}_R$ acting on a bullet of mass **m** for one half the period T_2 of each coning cycle produces a rightward horizontal bullet velocity increment ΔV_R given here in **feet per second per half-coning cycle** as:

$\Delta V_R = \{F_L\}_R*T_2/(2*m) = \{F_L\}_R/[2*m*f_2(t)] = (\pi/m)*\{F_L\}_R/\omega_2(t)$ (30)

Where **m** is the mass of the bullet expressed in **slugs**. Here, **m = 175.16/(7000*g) = 0.00077774 slugs**. We are using **g = 32.174 feet per second per second** for the standard effective "acceleration of gravity" on or near the surface of our rotating earth.

These rightward velocity increments ΔV_R accumulate (sum) from **zero** at **t = 0** for each *half coning cycle* which occurs from launch to time **t** to form the horizontally rightward velocity $V_R(t)$ of the CG of the bullet which is caused aerodynamically by the yaw of repose angle $\beta_R(t)$.

The incremental rightward horizontal spin-drift of the bullet, **ΔSD** in feet, during one particular half-coning cycle $T_2/2$ is then:

$\Delta SD(t) = V_R(t)*T_2/2 = V_R(t)/[2*f_2(t)] = \pi*V_R(t)/\omega_2(t)$ (31)

The horizontal spin-drift **SD(t)** at time **t** is then found by *summing* these incremental displacements **ΔSD(t)** for each *half coning cycle* starting with **zero** at **t = 0**. Our subject M118LR bullet experiences **87** complete **half-coning cycles** during its flight to **1000 yards.**

Numerical integration of **ΔSD(t)** using PRODAS data for each millisecond of the simulated "no wind" flight yields **SD(1.6923 sec) = 9.7019 inches.** PRODAS itself calculates a total drift of **9.5407 inches** at **1000 yards.** The PRODAS drift includes the horizontal component of the minimal coning motion of the spinning bullet.

This level of agreement verifies our aeroballistic analysis of the causes of spin-drift.

Analytic Calculation of the Spin Drift at the Target

If we formulate a reliable estimation of the scale factor **ScF** for any given bullet in any given firing conditions, this scale factor **ScF** can then be used together with a reliably calculated value of that bullet's *DROP from the bore axis at the target* to calculate analytically the spin-drift **SD(t)** at the target for any given rifle bullet in any firing conditions according to **Eq. 2**:

SD(t) = -ScF*DROP(t) **(2)**

We know from examination of available "constant crosswind" PRODAS 6-DoF simulations that the scale factor **ScF** needs to be **0.0219685** for the "constant no-wind" (minimum coning angle) runs, and **0.0222219 (or 1.154 percent larger)** for the somewhat more realistic "constant 10 MPH crosswind" runs for this example M118LR bullet fired in these simulated conditions to **1000 yards**.

The unrealistic "constant no-wind" PRODAS case represents the *minimum possible* **SD(t)** values for this bullet fired at this muzzle velocity and spin-rate in this atmosphere and flying with the minimum possible coning motion throughout its ballistic flight.

Most *dynamically stable* rifle bullets fired outdoors at long ranges will likely suffer only the minimal **1.154 percent** increased "constant 10 MPH crosswind" type of spin-drift **SD(t)** at long ranges.

However, a *dynamically unstable* rifle bullet, such as the infamous mid-range 30-caliber 168-grain Sierra International, might experience about **5 percent** greater spin-drift (**SD**) when fired to long ranges (up to 800 meters) through *any* non-zero crosswinds.

Estimating the Ratio of Second Moments of Inertia for Rifle Bullets

We need an accurate estimation of the ratio **Iy/Ix** for our rifle bullet so that we can find the coning rate **ω_2(t)** of that bullet at any time **t** during its flight from Tri-Cyclic Theory.

We will use the *reference diameter* for our subject bullet **d(in inches) = 1.00 calibers** as the distance metric throughout these calculations.

Input parameters are needed which describe the bullet.

We need the weight **Wt** of the bullet in grains, and we need the average density **ρ_P** of the bullet based on its type of construction: **2235.6 grains/cubic inch** for monolithic copper bullets; **2681 gr/in^3** for a thick-jacketed, lead-alloy-cored bullet having no appreciable hollow cavities; **2750 gr/in^3** for a thin-jacketed, pure-lead-cored match bullet; and **2120 gr/in^3** for monolithic bullets constructed of C360 brass.

We need the actual length **L** of the bullet in calibers. We need the actual length **LN** of the nose of the bullet in calibers. We need the diameter of the meplat **DM** at the front of the bullet in calibers. We also need the **RT/R** circular-arc head-shape design ratio for the ogive (nose) of the bullet, termed **RTR** here.

We then calculate the generating radius **RT** for a tangent ogive for this bullet, the full length **LFT** of a pointed *tangent ogive*, the full length **LFC** of a *conical ogive*, and the full nose length **LFN** of this bullet's actual ogive shape if it went all the way to a pointed tip.

$$RT = [LN^2 + ((1 - DM)/2)^2]/(1 - DM) \quad (32)$$

$$LFT = SQRT(RT - 0.25) \quad (33)$$

$$LFC = LN/(1 - DM) \quad (34)$$

$$LFN = LFT*RTR + LFC*(1 - RTR) \quad (35)$$

We then calculate The (full-ogive) total length **LL** of the bullet in calibers and shape factor **h** describing a cone-on-cylinder model of this rifle bullet:

$$LL = L - LN + LFN$$

$$h = LFN/LL \quad (36)$$

We now calculate the weight **Wtcalc** in grains for the cone-on-cylinder model of this bullet:

$$Wtcalc = (\pi/4)*\rho_P*d^3 *LL*(1 - 2*h/3) \quad (37)$$

We evaluate the polynomial **f_1(LL,h)** as:

$$f_1(LL,h) = 15 - 12*h + LL^2 *(60 - 160*h + 180*h^2 - 96*h^3 + 19*h^4)/(3 - 2*h). \quad (38)$$

We are now ready to calculate the ratio of the second moments of this bullet about its crossed principal axes:

$Iy/Ix = (Wt/Wtcalc)^{0.894} * f_1(LL,h)/[30*(1 - 4*h/5)]$ (39)

This estimator matches within **1 percent** the **Iy/Ix** ratios calculated by numerical integration for many different solid monolithic rifle bullet designs. Applying this estimator to data for the old 30-caliber 168-grain Sierra International bullet [from McCoy, page 217 MEB] yields an **Iy/Ix** ratio of **7.7748** or **4.48 percent** greater than the value **7.4413** reported by McCoy. This over-estimation is to be expected due to the significant hollow cavity within the nose of that bullet.

Estimating the Spin Drift Scale Factor ScF

With all this in mind, we formulate an estimator for an **ScF** value for the "constant 10 MPH crosswind" type of coning motion of our example bullet which can be duplicated for any other *dynamically stable* rifle bullet in any likely firing conditions.

In flat firing, we can formulate the scale factor **ScF** in terms of the *ratio* of 1) the horizontal aerodynamic lift-force acting on the flying bullet due to its yaw of repose β_R as that bullet nears its long-range target to 2) the vertically downward-acting weight of that bullet. In this manner, we can formulate **ScF** for any given bullet and likely wind conditions as:

$$ScF = 1.01154*0.383703*[q(t)*S]*Sin[\beta_R(t)]*CL_\beta(t)/Wt$$

$$ScF = 0.388132*[q(t)*S]*\beta_R(t)*CL_\beta(t)/Wt \tag{40}$$

with **Wt = 175.16/7000** representing the weight of this example M118LR bullet given in pounds-force **lbf.**

Here again, as in the earlier simplified formulation of $\Delta\Phi$, we are ignoring the upward force on the free-flying bullet caused by the cross-bore component of its aeroballistic drag force. The force offset effects of these two simplifications cancel out here in forming this ratio for evaluating **ScF**.

We define the scale factor **ScF** as the ratio of the magnitudes of the net rightward horizontal and downward vertical forces acting upon the flying bullet as a *free body*:

$$ScF = \{F\}_H/\{F\}_V \tag{41}$$

From **Eq. 29** above, the net rightward horizontal force is $\{F\}_H$:

$$\{F_L\}_R \approx q(t)*S*CL_\beta(t)*\beta_R(t) - q(t)*S*CD(t)*\beta_T(t) \tag{29}$$

The corresponding net downward vertical force $\{F\}_V$ is given by:

$$\{F\}_V = Wt - \Delta\Phi*F_D = Wt - \Delta\Phi*q(t)*S*CD(t) \tag{42}$$

Substituting these expressions into **Eq. 41** and simplifying by utilizing **Eq. 3** above,

$$ScF*\{F\}_V = \{F\}_H = q(t)*S*CL_\beta(t)*\beta_R(t) - q(t)*S*CD(t)*\beta_T(t) = \tag{43}$$

$$ScF*\{F\}_V = ScF*Wt - (ScF*\Delta\Phi)*q(t)*S*CD(t) = ScF*Wt - q(t)*S*CD(t)*\beta_T(t) \tag{44}$$

After adding the small quantity $q(t)*S*CD(t)*\beta_T(t)$ to these equal expressions, **Eq. 43** and **Eq. 44**, we have:

$$ScF*\{F\}_V + q(t)*S*CD(t)*\beta_T(t) = q(t)*S*CL_\beta(t)*\beta_R(t) = ScF*Wt$$

$$ScF= q(t)*S*CL_\beta(t)*\beta_R(t)/Wt \tag{45}$$

which is the expression which we will actually evaluate for **ScF** at time **T** far downrange.

The initial constants (**0.383703** and **0.388132**) have been empirically determined from several PRODAS runs and should be the same for any *dynamically stable* rifle bullet in any firing conditions likely to be encountered in long-range shooting. The PRODAS runs show the M118LR bullet to be dynamically stable.

These PRODAS simulations, together with the classic formulation for the yaw of repose angle **$\beta_R(t)$**, indicate that the scale factor **ScF** might need to be increased by about **5 percent** for *dynamically unstable* bullets which will fly with significant coning angles throughout their flight when fired through *any crosswinds at all*.

Each of these functions of time **t** should be evaluated at the time **T** when the bullet has slowed to an airspeed of **1340 feet per second** (or approximately **Mach 1.20**, depending upon ambient conditions).

This flight time **T** and the flight distance **Rg** at which it occurs are completely independent of the actual range to the target. The time **T** and range **Rg** to **1340 fps** *airspeed* can be determined by using any current 3-DoF point-mass trajectory calculator.

The airflow over any good long-range rifle bullet should remain safely above the turbulent transonic region at this **1340 fps** airspeed in almost any reasonable atmospheric conditions. The more "aerodynamic" of our lowest-drag long-range rifle bullet designs will not encounter transonic buffeting until they slow to about **Mach 1.10** airspeed. The needed coefficient of lift **CL_β** is particularly difficult to estimate for any bullet in the transonic airspeed regime.

Most experienced long-range riflemen select their shooting equipment so that whenever possible their bullets will impact the target at airspeeds above **Mach 1.2**.

For similar best-accuracy reasons, we base our calculation of **ScF** upon bullet data at **1340 fps** airspeed regardless of the actual range to the intended target.

We calculate the potential drag-force **q(T)*S** using the calculated density **ρ** of the ambient atmosphere in **slugs per cubic foot**, and the airspeed **V(T) = 1340 feet per second**:

$$q(T)*S = (\pi*d^2/4)*(\rho/2)*(1340\ \text{fps})^2 \qquad (46)$$

This potential drag-force value should be about **1.1 lbf** for a **30-caliber** bullet at this airspeed depending on air density. The potential drag-force at **1340 fps** varies most strongly with the square of the caliber **d** of the bullet (in feet).

The analytic estimate of **$\beta_R(T)$** is calculated per **Eq. 26** above with **t = T** and **V(T) = 1340 fps**. With these simplifications **Eq. 26** becomes:

$$\beta_R(T) = -g*\{\exp[-(k\omega + k_V)] - 1\}/[f_2(0)*V(0)*(k\omega + k_V)] \qquad (47)$$

If we know the *initial* gyroscopic stability **Sg** of the bullet, we can calculate the initial Stability Ratio **R** of its epicyclic rates f_1/f_2 from:

$$R = 2*\{Sg + SQRT[Sg*(Sg - 1)]\} - 1 \qquad (48)$$

The *initial* coning rate $f_2(0)$ in hertz can then be found from Tri-Cyclic Theory as:

$$f_2(0) = V(0)/[Tw*(Iy/Ix)*(R + 1)] \qquad (49)$$

where

Tw = Absolute value of the Twist Rate of barrel in feet per turn.

Iy/Ix = Ratio of transverse to axial second moments of inertia for this bullet as estimated above.

Substituting back into **Eq. 47**, we have:

$$\beta_R(T) = -g*Tw*(Iy/Ix)*(R + 1)*\{exp[-(k_\omega + k_V)] - 1\}/[V^2(0)*(k_\omega + k_V)] \qquad (50)$$

where

V(0) = Launch velocity of this particular bullet in feet per second. [**V(0)** is assumed to exceed **Mach 2.0**]

$k_\omega = -(0.585 + 0.0321/d)*T$, as approximated earlier and used here for any long-range bullet, and

$k_V = \ln[1340/V(0)]$.

This value $\beta_R(T)$ in radians is an estimate of the yaw of repose angle for this bullet where it slows to an airspeed of **1340 fps**.

The $CL_\beta(T)$ value is estimated based on an estimate of the initial $CL_\beta(0)$ for the Mach-speed of the bullet at launch (here **Mach 2.3289**) evaluated from Robert L. McCoy's INTLIFT program for the nose-length effect, but using our own boat-tail effect lift reduction for these long-range bullets.

We multiply McCoy's nose-length estimated **CL** by the square root of **0.2720/BC7** for each bullet, reasoning that about half of any differing drag for bullets having higher or lower ballistic coefficients **BC7** (relative to the G7 Reference Projectile) than our example M118LR bullet is due to having a more or less effective boat-tail design.

If a more reliable **BC1** value (relative to the G1 Reference Projectile) is available for your rifle bullet, use the square root of **0.5310/BC1** for this estimated **CL** adjustment for variations in bullet drag.

The full nose-length (**LFN**) for the 30-caliber M118LR bullet is **2.5955 calibers**. The initial coefficient of lift **$CL_\beta(0)$** at this **Mach 2.3289** airspeed calculates to **2.720** using our adjusted INTLIFT estimate.

Very slightly scaling McCoy's published lift curve for the well-studied 30-caliber 168-grain Sierra International bullet to this lift coefficient **2.720** at this **Mach 2.3289** airspeed yields a coefficient of lift **$CL_\beta(T)$** of **1.877** at **Mach 1.20** in this standard sea-level ICAO atmosphere.

The bullet's coefficient of drag **CD_0** determines its time-rate of decay in Mach-speed. The supersonic lift-to-drag ratio **F_L/F_D** for any given angle-of-attack tends to be an invariant aerodynamic characteristic of each basic bullet shape.

Since the coefficients of lift and drag are highly correlated at any given Mach-speed over the population of long-range rifle bullets, the same exponential time-decay coefficient:

$$k_L = \ln[CL_\beta(T)/ CL_\beta(0)] = -0.3711 \quad (51)$$

can be used in propagating the coefficients of lift **$CL_\beta(T)$** for any long-range rifle bullets of interest here.

We propagate this *initial* coefficient of lift **$CL_\beta(0)$** estimate forward to its value at time **T** as:

$$CL_\beta(T) = CL_\beta(0)*\exp\{-0.3711*[V(0)/2600\text{ fps}]^2 *(1.430\text{ sec}/T)\} \quad (52)$$

The coefficient of lift **$CL_\beta(T)$** for any very-low-drag (VLD) or ultra-low-drag (ULD) long-range rifle bullet should be smaller than **1.90** at this airspeed of **1340 fps**. Bullets designed for lower aerodynamic drag will also produce less aerodynamic lift. Conversely, one cannot produce more lift without also increasing drag in aerodynamics.

The exponential propagation function [**Eq. 52**] estimates a larger fraction of the initial coefficient of lift **$CL_\beta(0)$** remaining at **1340 fps** airspeed when the time-of-flight **T** ***to that airspeed*** is increased due to firing a higher-drag bullet, but the initial velocity correction factor **$[V(0)/2600\text{ fps}]^2$** prevents this increase when time-of-flight **T** to **1340 fps** increases simply due to firing that same bullet with a higher muzzle velocity **V(0)**.

That is, if the ***same bullet*** is fired at different muzzle velocities, its estimated coefficient of lift **$CL_\beta(T)$** when it has slowed to an airspeed of **1340 fps** should remain the same.

The muzzle velocity **V(0)** is assumed to exceed **Mach 2**. This coefficient of lift propagation yields the expected **$CL_\beta(T) = 1.8769$** at **1340 fps** for the M118LR bullet, and varies by less than **1 percent** over any reasonable launch speeds **V(0)** for this one bullet type.

The scale factor **ScF** is now calculated from **Eq. 40** using the values of the time-functions at time **T** as calculated in **Eq. 46**, **Eq. 50**, and **Eq. 52** above:

$$ScF = 0.388132*[q(T)*S]*\beta_R(T)*CL_\beta(T)/Wt \quad (53)$$

where

0.388132 = An empirically determined constant (from PRODAS data) for all firings of "normally coning" *dynamically stable* rifle bullets through any non-zero, "reasonably steady" (non-diabolical) crosswinds.

This constant is numerically necessary for several likely reasons, among them that the driving horizontal lift-force $\mathbf{F_L[t, \beta_R(t)]}$ is actually attributable only to the dynamic horizontal tracking error attitude angle $\boldsymbol{\varepsilon_H(t)}$ instead of the entire yaw-of-repose angle $\boldsymbol{\beta_R(t)}$.

If we might be slightly misestimating the yaw of repose angle $\boldsymbol{\beta_R(T)}$ or coefficient of lift $\mathbf{CL_\beta(T)}$ used here in any systematic ways for these minimal-coning-motion "constant wind" 6-DoF flight simulations, the empirically-determined initial constant factor **0.388132**, from that same PRODAS data, tends to absorb any net systematic difference.

Calculating the Spin Drift at the Target

The spin-drift **SD(tof)** at the *target distance* is calculated from **Eq. 2** above using the invariant Scale Factor **ScF**, as calculated in **Eq. 53** above for the bullet slowed to **1340 fps**, and the total **DROP** from the axis of the bore for the actual time-of-flight (**tof**) to the target:

$$SD(tof) = -ScF*DROP(tof) \tag{54}$$

The Spin-Drift **SD** at the target is calculated here in **Eq. 54** in the same *distance* or *angular units* in which the bullet's **DROP** from the bore axis is given. Again, the proper algebraic sign depends upon coordinate system conventions and the sense of the bullet's spin rotation.

The bullet **DROP** and time-of-flight (**tof**) to the target are accurately calculated in many existing 3-DoF point-mass trajectory propagators. After all, the accurate calculation of bullet **DROP** at the target distance is the basic figure of merit for these software aids.

The time-of-flight (**tof**) to the target is used in the Litz **SD** estimator and is nice to know even if we do not actually use it explicitly in these calculations.

If your particular trajectory propagation program does not directly output "drop from bore axis" data, you can usually "fake" it into doing so by setting your scope height equal zero, setting the angle-of-fire accurately equal to that of the anticipated shot, setting the rifle's "zero range" equal to some minimum distance (ideally zero, but perhaps 5 or 10 yards if made necessary by input limit constraints), and by specifying that the trajectory calculations go out to the target's known or measured range.

In other words, we want to calculate the **DROP** from the bore axis at the target distance as if we were "bore-sighted" on that long-range target.

The smoothed spin-drift reported by PRODAS at 1000 yards for "constant zero-wind" simulations with this 175.16-grain M118LR bullet fired in these conditions is **9.5407 inches**. The spin-drift **SD** at 1000 yards estimated via this algorithm using PRODAS data values (and without the factor of **1.01154** increase in **ScF**) is **9.5635 inches**.

Comparing the two results millisecond-by-millisecond, throughout the flight of **1692.3 milliseconds**, yields a mean difference of **0.0043 inches,** with a population standard deviation of **0.0207 inches**.

This level of agreement between our analytical estimator for spin-drift for each millisecond and the PRODAS numerical (non-analytical) simulation results is rather astonishing. The rounding error for drop and drift data given in angular units in the PRODAS report format is **0.180 inches** at 1000 yards, and we are not even estimating the horizontal component of the bullet's small coning motion which is included in the PRODAS drift data.

The agreement of this spin-drift **SD** estimator with PRODAS "constant 10 MPH crosswind" runs is also excellent when the Scale Factor **ScF** includes the **1.154 percent** increase as formulated above.

This **1.154-percent-augmented** version of the **ScF** estimator in **Eq. 48** should be calculated for outdoor firing of any other *dynamically stable* rifle bullets.

Summary

I. We fit an exponential tangent angle function $\beta_T(t)$ to extracted velocity-ratio data from a PRODAS simulation which minimizes the epicyclic swerve complications in measuring the yaw of repose angle $\beta_R(t)$. We discovered that the spin-drift **SD** at long range is affected slightly (about **5 percent**) by the magnitude $\alpha(t)$ of coning motion experienced by the bullet en route to the target, with consistently larger coning angles $\alpha(t)$ producing slightly more spin-drift $SD(t)$.

II. We define the horizontal and vertical direction dynamic tracking error angles, ε_H and ε_V respectively, which should appear in a ballistician's wind-axes plots resulting from 6-DoF flight simulations. Just as with the flight path angle Φ, the yaw of repose angle $\beta_R(t)$ logically should not appear in those wind-axes plots which reference as their origin the $+V$ direction of the projectile's mean velocity vector V, which is always tangent to its 3-dimensional mean trajectory. We provide an analytic formulation in **Eq. 3** for $\beta_T(t)$, the horizontal tangent angle, which logically should be subtracted from the bullet's spin-axis yaw attitude data before plotting.

III. We explain the aerodynamic causes of yaw of repose and spin-drift and numerically verify those explanations using data from PRODAS 6-DoF simulations together with the principles of linear aeroballistic theory.

IV. We reformulate the classic aeroballistic yaw of repose angle as $\beta_R = \pi PG/M$, which holds for a significantly coning bullet with $\alpha(t) >> \Delta\Phi$ throughout its flight. Furthermore, $\beta_R(0) = 0.00$ at launch by definition. For a minimally coning bullet with $\alpha(t) \gtrsim \Delta\Phi$, $\beta_R = \beta_T + \varepsilon_H \approx 0.95\pi PG/M$.

V. We formulated an accurate analytic estimator for the ratio Iy/Ix of the second moments of inertia for any long-range rifle bullet so that the sum of its two epicyclic rates $(\omega_1 + \omega_2)$ can be calculated via Tri-Cyclic Theory from the circular spin-rate ω of the bullet (in radians per second) remaining at any time **t** during its flight. We noted that the spin-rate $\omega(t)$ decreases very nearly exponentially with time **t** for modern rifle bullets:

$\omega(t) \approx \omega(0)*\exp\{[(-0.0321/d(\text{inches})]*t\}$.

VI. We note that in flat firing the spin-drift displacement **SD** of the bullet at any long range is essentially an invariant scale factor **ScF** times the bullet's drop distance from the projected bore axis at that range. The scale factor **ScF** runs about **1.0** to **2.3 percent** for the various long-range rifle bullets in typical flat firing. That bullet's drop from the axis of the bore is accurately computed in any 3-DoF trajectory propagation program. This same scale factor **ScF** defines the ratio of the horizontal and vertical angular deviations of the tangent to the ***mean trajectory*** from the axis of the bore at firing time (when $t = 0$). The angular deviation in the horizontal plane $\beta_T(t)$ is always equal to the Scale Factor **ScF** times the vertical-direction deviation $\Delta\Phi_{Total}(t) = \Phi(t) - \Phi(0)$.

VII. We present an analytic calculation of that invariant scale factor **ScF** so that an accurate and reliable calculation of spin-drift **SD(t)** can be computed for any long-range rifle bullet flat-fired in any likely conditions without relying upon 6-DoF simulations. This dimensionless Scale Factor **ScF** can also be used as part of a collection **K** of invariant values from **Eq. 40** such that the yaw of repose angle $\beta_R(t)$ can be calculated for any flight time **t** as:

$$\beta_R(t) = K/[V^2(t)*CL_\beta(t)] \qquad (55)$$

with

$$K = (ScF/0.388132)*[Wt/(\rho*S/2)] \qquad (56)$$

This formulation of $\beta_R(t)$ is very similar to the classic formulation for $\beta_R(t)$, and this formulation also does ***not*** evaluate to zero at $t = 0$. This calculated non-zero initial yaw of repose attitude angle $\beta_R(0) \approx$ **0.130 milliradians** is just the initial yaw attitude which would be required to produce a hypothetical horizontal lift force of **ScF*Wt** at muzzle velocity **V(0)**. Of course, no such side-force exists at bullet launch.

Example Calculations of Iy/Ix

These parameters and calculations are needed to determine the crucially important ratio of the bullet's second moments of inertia about its crossed principal axes **Iy/Ix**.

Four different 30-caliber rifle bullets are selected in addition to our example M118LR bullet in these parallel (spreadsheet) calculations for variety and based on availability of bullet measurements for estimating **Iy/Ix** ratios. Two of the additional 30-caliber rifle bullets are included because they were tested in "drift firings" by Bryan Litz.

The obsolete 168-grain Sierra International bullet (for which McCoy supplies the needed data) is similar to their current, improved 30-caliber 168-grain MatchKing (SMK). We tabulated the calculations of **Iy/Ix** because this ratio was measured and reported by McCoy.

We have several PRODAS runs for the bullet used in 2011 in the US Army M118LR 7.62 mm NATO Special Ball ammunition. The 175.16-grain M118LR bullet used by PRODAS has an **Iy/Ix** ratio which can be determined very accurately from the PRODAS reports. We are using reasonably estimated bullet shape parameters scaled from images which produce approximately that PRODAS calculated **Iy/Ix** value in lieu of the actual bullet shape data on the M118LR bullet until such data can be obtained.

The 173-grain solid (monolithic) brass Ultra-Low-Drag (ULD) bullet design has not yet been tested, but its numerical design description allows accurate modeling of its flight characteristics using McCoy's aeroballistic estimators. We calculated its mass characteristics, including its **Iy/Ix** ratio, using accurate numerical integration.

The dimensional data on the Berger 175-grain Open-Tip Match (OTM) Tactical bullet and their 185-grain Long Range Boat Tail (LRBT) bullets, as well as the test conditions during their 1000-yard drift firings, were taken from Bryan Litz's publications.

An **Iy/Ix** ratio of **7.4413** is published by McCoy for the old 168-grain Sierra International bullet. Our estimate of **7.7748** is **4.48 percent** larger than McCoy's measured value. This over-estimation is to be expected given that bullet's significant hollow nose cavity.

The target **Iy/Ix** ratio of **13.4733** for the new **173-grain** monolithic brass ULD bullet was calculated by numerical integration of its elements of mass. Our estimated value here of **13.4975** is just **0.180 percent** larger than this value.

The data used here for the two Berger 30-caliber bullets selected by Bryan Litz in his drift firing experiments at 1000 yards are taken from his publications. Long-range drift firings are a traditional method for measuring horizontal spin-drift. No information is available concerning their **Iy/Ix** ratios.

30-Caliber Example Bullets:	168-gr International	175.16-gr M118LR	173-gr ULD(SB)	175-gr Berger Tactical	185-gr Berger LR-BT
Reference Diameter (inches)	**0.3080**	**0.3080**	**0.3002**	**0.3080**	**0.3080**
Bullet Length L (cal)	**3.9800**	**4.4000**	**5.4368**	**4.1169**	**4.3929**
Nose Length LN (cal)	**2.2600**	**2.4500**	**2.8368**	**2.3701**	**2.5747**
Diameter of Meplat DM (cal)	**0.2500**	**0.2175**	**0.1000**	**0.1948**	**0.2013**
Length of Boat-Tail LBT (cal)	**0.5100**	**0.6000**	**0.7012**	**0.6331**	**0.5844**
Diameter of Base DB (cal)	**0.7645**	**0.8000**	**0.8420**	**0.8409**	**0.8182**
Ratio of Ogive Generating Radii RT/R	**0.9000**	**1.0000**	**0.5000**	**0.9000**	**0.9500**
Rho-P, Bullet Density in grains/cu. in.	**2750**	**2600**	**2128**	**2750**	**2750**
Rho-P, Ave. Specific Gravity (gm/cc):	**10.8742**	**10.2811**	**8.4147**	**10.8742**	**10.8742**
Wt, Bullet Weight in grains	**168**	**175.16**	**173**	**175**	**185**
Calc. Tangent Ogive Radius RT (cal)	**6.9976**	**7.8666**	**9.1666**	**7.1779**	**8.4993**
Calc. Full Tangent Ogive Length LFT (cal)	**2.5976**	**2.7598**	**2.9861**	**2.6321**	**2.8722**
Calc. Full Conical Nose Length LFC (cal)	**3.0133**	**3.1310**	**3.1520**	**2.9435**	**3.2236**
Calc. Full Nose Length LFN (cal)	**2.6392**	**2.7598**	**3.0690**	**2.6632**	**2.8897**
Length LL with LFN (cal)	**4.3592**	**4.7098**	**5.6690**	**4.4100**	**4.7079**
h, Ratio LFN/LL	**0.6054**	**0.5860**	**0.5414**	**0.6039**	**0.6138**
Wtcalc, Calc Wt in grains	**164.0601**	**171.2331**	**163.8186**	**166.2542**	**175.5264**
f1(LL,h)	**117.7480**	**141.1386**	**218.6318**	**120.6627**	**133.9879**

Cone-on-Cylinder Estimated Iy/Ix=	**7.7748**	**9.0376**	**13.4975**	**8.1466**	**9.1976**
Target Iy/Ix	**7.4413**	**9.0673**	**13.4733**		
Iy/Ix Error:	**0.3335**	**-0.0297**	**0.0242**		
Percent Error:	**4.481**	**-0.328**	**0.180**		

Example Calculations of Spin Drift

The remaining parameters needed to calculate yaw of repose β_R and spin-drift **SD** are calculated for these same five example bullets in another spreadsheet shown below. A 3-DoF trajectory program was used to compute the time-of-flight (**tof**) and flight distance to an airspeed of **1340 FPS** and **tof** to a 1000-yard target for both the 168-grain SMK bullet and the new 173-grain ULD bullet. PRODAS trajectory data was used for the 175.16-grain M118LR bullet.

The initial gyroscopic stability factor **Sg** was taken from McCoy for the old 168-grain Sierra International bullet, and **Sg** is calculated using McCoy's McGYRO program for the new 173-grain ULD bullet.

PRODAS reports the **Sg**-value for each millisecond of the flight of the M118LR bullet, but we just used their initial value. Bryan Litz gives the initial **Sg** values for the two Berger bullets used in his drift firings.

The ULD bullet is a dual-diameter design with the base of the ogive measuring **0.3002 inches** in diameter (**1.0-calibers** for this bullet design). It has a rear driving-band measuring **0.3082 inches** in diameter (or **1.02665 calibers**). The midpoint (CG) of the rear driving-band is located **1.6 calibers** behind the base of its 3-caliber secant ogive, and the width of this driving band is **0.6 calibers**.

Our calculated yaw of repose angles β_R for the first three example bullets when each has slowed to an airspeed of **1340 FPS** shows an interesting progression.

The estimated yaw of repose angles β_R of the three trajectories at the **1340 FPS** airspeed points are **0.471231 milliradians** for the obsolete 168-grain International bullet at **816 yards** downrange, and **0.693417 milliradians** at **888.5 yards** downrange for the M118LR bullet, but just **0.434382 milliradians** for the new 173-grain monolithic brass ULD bullet fired at **3200 fps**, and this occurs way beyond the 1000-yard target at **1457 yards** downrange.

The assumed **3200 fps** muzzle velocity of this new ULD bullet is based on firing it from a 300 Remington UltraMag cartridge. Each of the other example 30-caliber bullets is assumed to be fired from a much less powerful 7.62 mm NATO or 308 Winchester cartridge.

For comparison purposes the spin-drift **SD** at **1000 yards** is calculated in inches for each of our five example bullets using the **SD** estimator published by Bryan Litz:

$$SD(inches) = 1.25*(Sg + 1.2)*(tof)^{1.83} \qquad (57)$$

Our estimates of **SD** at 1000 yards are smaller than Bryan's estimates for each of these five example bullets. Our estimate of spin-drift **SD** at 1000 yards for the M118LR bullet of **9.5111 inches** matches the **SD** computed by PRODAS (**9.5407 inches**) quite closely (error: **-0.0296**

inches, or **-0.003 percent**). The Litz-estimated **SD** of **10.2791 inches** for this M118LR bullet at 1000 yards exceeds the PRODAS value by **0.7156 inches**, or **+7.483 percent**.

Our estimate of **6.8061 inches** of spin-drift **SD** at 1000 yards for the old 168-grain Sierra International bullet from a 12-inch twist barrel is approximately **2.344 inches** less than the **9.15 inches** shown graphically by McCoy in **Figure 9.8** of his MEB, and is **3.214 inches** less than the **10.020 inches** calculated by the Litz estimator for this bullet.

We expected our estimate to be **5 percent** (or **0.340 inches**) too small for this *dynamically unstable* bullet. We cannot readily explain the remainder of this difference. Perhaps we should concede that this formulation inherently assumes that the rifle bullet is *dynamically stable*.

Our estimate of **4.2983 inches** of spin-drift **SD** at 1000 yards for the radical new 173-grain monolithic brass ULD bullet design, versus the value of **5.1696 inches** calculated by the Litz estimator for this bullet, indicates the need for our more elaborate **SD** calculation in predicting the long-range flights of current and future ultra-low-lift rifle bullets, even when fired from faster twist-rate barrels.

The Litz spin-drift estimator is closer than our estimator to reported spin-drift values for the old 168-grain Sierra International bullet and for the Berger 175-grain OTM Tactical bullet.

Our estimator seems closer for the remaining three bullets, especially for the two very-low-drag (and correspondingly very-low-lift) bullets—the copper 173-grain ULD bullet and the Berger 185-grain Long Range Boat-Tail (LR-BT) bullet.

Of course, our predictive agreement with the PRODAS calculations for the M118LR bullet is best of all. We expect that if 6-DoF simulations could be run for the other four bullets, our estimator would match those results more closely.

We also suspect that Bryan's drift firing results would not match linear 6-DoF simulation results particularly well if they could be computed.

The aerodynamic responses of real rifle bullets are non-linear enough to affect the calculation of these small second-order effects. Real bullets are also subject to other types of aerodynamic jump phenomena in real firings—some of which might be at least partially systematic.

Spin-Drift Example Calculations:					
30-Caliber Example Bullets:	168-gr International	175.16-gr M118LR	173-gr ULD	175-gr Berger Tactical	185-gr Berger LR-BT
Bullet Length L (cal)	3.9800	4.4000	5.4368	4.1169	4.3929
Nose Length LN (cal)	2.2600	2.4500	2.8368	2.3701	2.5747
Diameter of Meplat DM (cal)	0.2500	0.2175	0.0808	0.1948	0.2013
Length of Boat-Tail LBT (cal)	0.5100	0.6000	0.7012	0.6331	0.5844
Diameter of Base DB (cal)	0.7645	0.8000	0.8420	0.8409	0.8182
Ratio of Ogive Generating Radii RT/R	0.9000	1.0000	0.5000	0.9000	0.9500
Calc. Tangent Ogive Radius RT (cal)	6.9976	7.8666	8.9846	7.1779	8.4993
Calc. Full Tangent Ogive Length LFT (cal)	2.5976	2.7598	2.9554	2.6321	2.8722
Calc. Full Conical Nose Length LFC (cal)	3.0133	3.1310	3.0862	2.9435	3.2236
Calc. Full Nose Length LFN (cal)	2.6392	2.7598	3.0208	2.6632	2.8897
V0=Launch velocity (FPS):	2800.00	2600.07	3200.00	2660.00	2630.00
Initial Mach-Speed	2.5079	2.3289	2.8662	2.4332	2.4057
Initial B-value	2.3000	2.1032	2.6861	2.2182	2.1880
Ballistic Coef (G1 Ref)	0.4260	0.5460	0.6290	0.5060	0.5530
Ballistic Coef (G7 Ref)	0.2180	0.2720	0.3220	0.2580	0.2830
INTLIFT CL(0)	3.1015	2.6759	2.5670	2.8145	2.6189
Time T to 1340 FPS (sec)	1.2723	1.4300	2.1070	1.3972	1.5030
Range at 1340 FPS Airspeed (yards)	816.00	888.50	1457.00	881.54	839.10
Est CL(T) at 1340 FPS:	1.9120	1.8463	1.7528	1.8913	1.8248

Twist Rate (inches/turn, RH)	12.0000	11.5000	8.2500	10.0000	10.0000
Initial Sg	1.7400	1.9400	1.5940	2.2400	1.9100
Initial Stability Ratio (R)	4.7494	5.5808	4.1341	6.8132	5.4567
Calculated (Iy/Ix) Ratio	7.7748	9.0376	13.4975	8.1466	9.1976
Initial Coning Rate f2 (hz)	62.6387	45.6182	67.1674	50.1486	53.1437
kv=LN(1340/V(0)	-0.73695	-0.66287	-0.87048	-0.68566	-0.67431
komega+kv	-1.61385	-1.64845	-2.32837	-1.64866	-1.71021
Beta-R at time T (mrad)	0.4572	0.6909	0.5954	0.6144	0.6098
Ref. Diam. (1.0 cal. in inches):	0.3080	0.3080	0.3002	0.3080	0.3080
Frontal Area at Base of Ogive S (square feet)	0.0005174	0.0005174	0.0004915	0.0005174	0.0005174
Potential Drag Force at 1340 fps (lbf)	1.1041	1.1041	1.0489	1.1137	1.1137
Bullet Weight (grains)	168.00	175.16	173.00	175.00	185.00
Bullet Weight (lbf)	0.02400	0.02502	0.02471	0.02500	0.02643
Calculated Scale Factor ScF	0.01561	0.02185	0.01719	0.02009	0.01820
DROP from Bore Axis at 1000 yds (inches)	436.0450	435.3450	250.0250	428.4970	414.8350
Time of Flight (tof) to 1000 yds (seconds)	1.7300	1.6923	1.2390	1.6870	1.6400
Remaining Velocity at 1000 yds (FPS)	1145.00	1213.99	1836.00	1197.00	1278.00
Calculated 1000-yd Spin-Drift (inches rightward)	6.8061	9.5111	4.2983	8.6095	7.5496
SD from McCoy Figure 9.8	9.1500				
SD from PRODAS runs		9.5407			

SD (inches) from Drift Firings				**11.4000**	**6.7000**
Litz Est. Spin-Drift (inches rightward)	**10.0203**	**10.2791**	**5.1696**	**11.1967**	**9.6125**
Litz Est. Spin-Drift Minus Our Calc. SD (inches)	**3.2142**	**0.7679**	**0.8713**	**2.5871**	**2.0628**

Sensitivity Analysis & Model comparisons

Sensitivity analysis was significant in studying and assessing the uncertainty in the output of our model, which can be attributed to different sources of error of the input parameters.

Sensitivity analysis is an integral part of model development and involves analytical examination of input parameters to aid in model validation and provide guidance for future research.

We used it to determine how different values of one or more independent variables, impact a particular dependent variable under a given set of conditions.

In other words, it helped us to investigate the robustness of the model predictions and to explore the impact of varying input assumptions.

We chose to set on what is known as local (sampled) sensitivity analysis, which is derivative based (numerical or analytical). The use of this technique is the assessment of the local impact of input factors' variation on model response by concentrating on the sensitivity in vicinity of a set of factor values.

Such sensitivity is often evaluated through 2-dimensional gradients or partial derivatives of the output functions at these factor values, (the values of other input factors are kept constant) when studying the local sensitivity of a given input factor.

One of the critical objectives was to stress-test the model as well as to study its fidelity to known experimental and model-based 6-DoF runs.

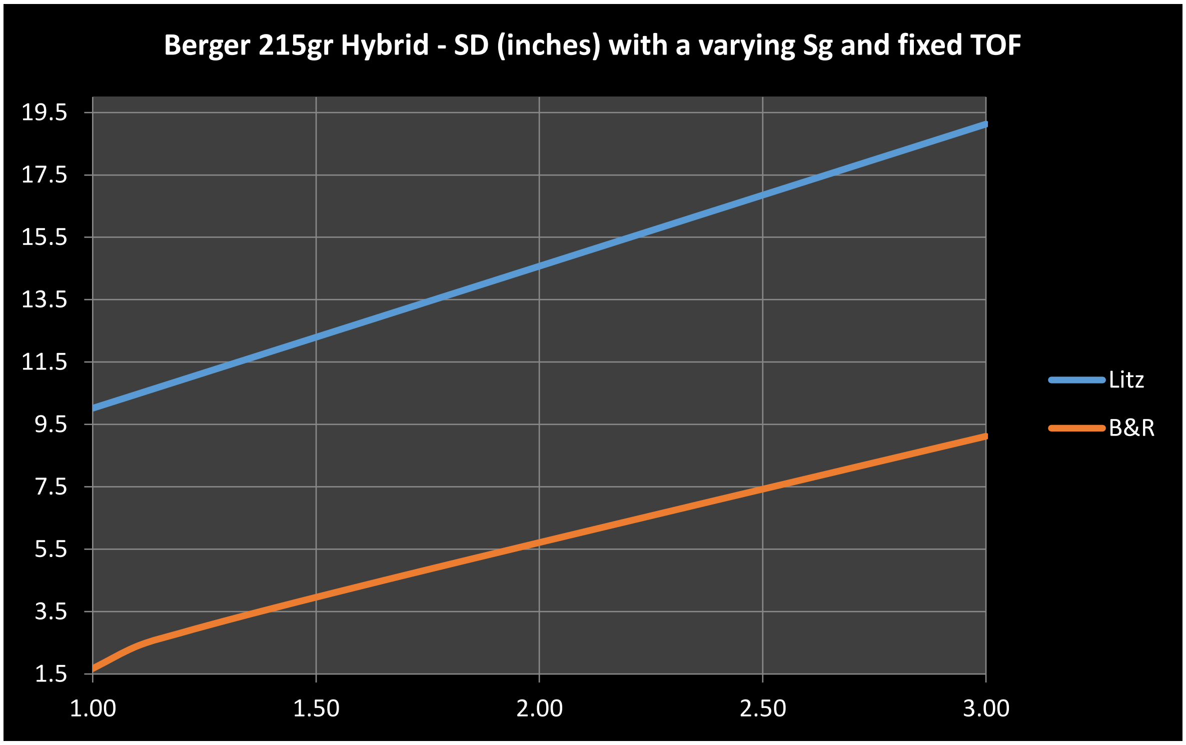
Unfortunately there are many results and accompanying charts to add, but in order to make the document more manageable we've chosen to include only a pair of them, which are significant in terms of reliability of the underlying numerical algorithm.

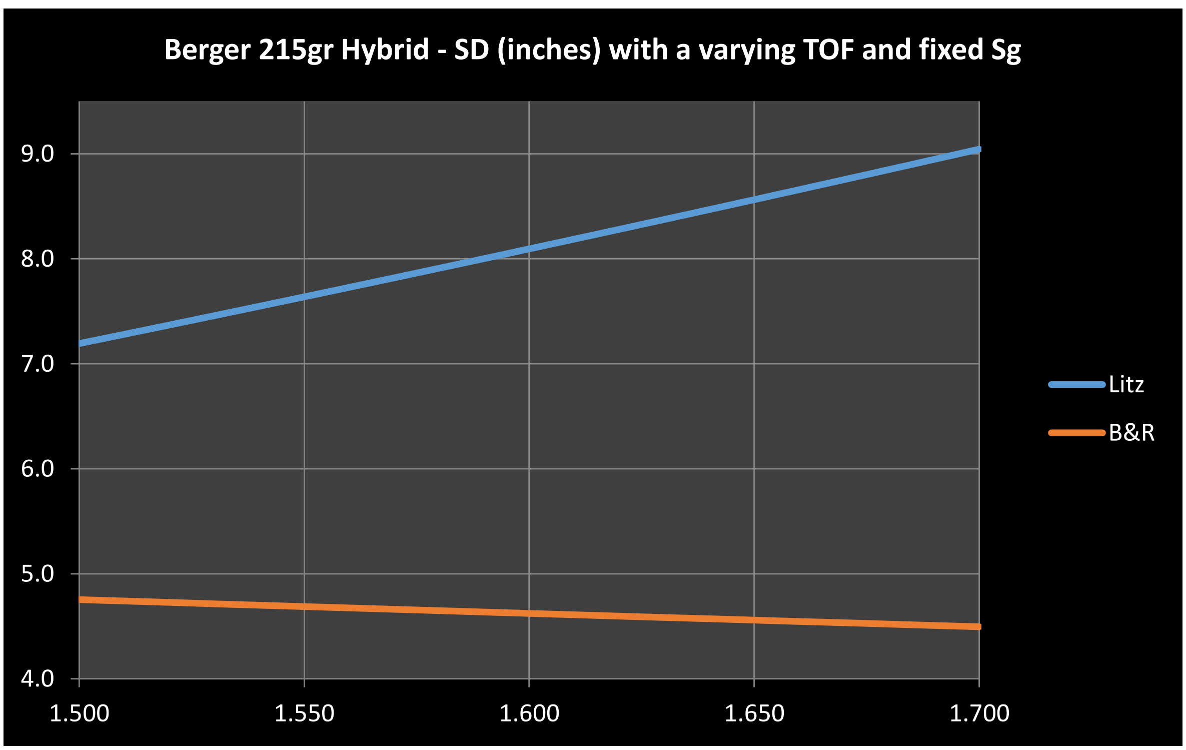
The following charts compare the outputs of three models to estimate **SD**, namely Hornady 4-DoF, Litz and the B&R method presented in this paper.

In the case of Hornady's 4-DoF, the reader must take into consideration that three major variables, **Sg**, **DROP** and **ToF**, are different than the ones used to calculate the B&R and Litz outputs because it produces different **DROP** and **ToF** values as well as a varying **Sg**.

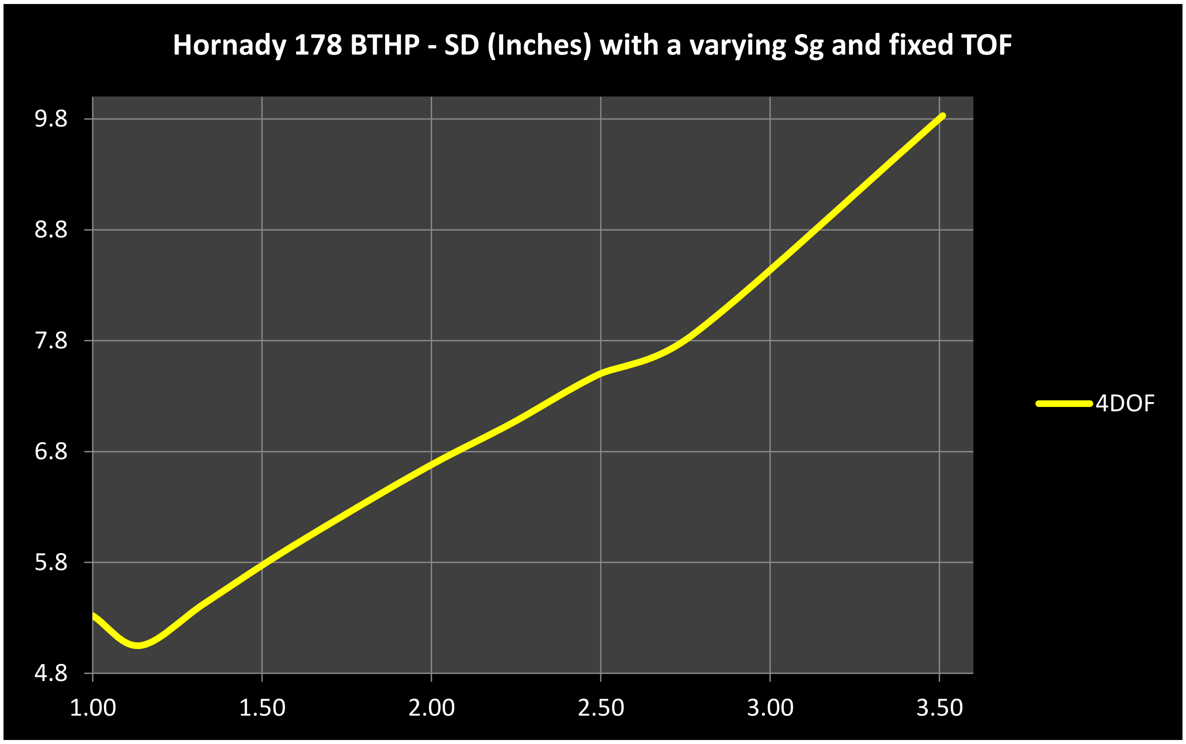
On the other hand, all **DROP** and **ToF** figures are the same for both Litz and B&R, and were calculated with a common 3-DoF point mass software with a fixed-muzzle-only **Sg** based on Miller' rule. Indeed neither model is intended to work with a progressively increasing static stability.

As can be easily seen, the response to a varying **Sg** with a fixed **ToF** is clearly linear for both models. Same behavior for a varying **ToF** with a fixed **Sg**. Bear in mind that the variation ranges are quite narrow, which is the normal and expected uncertainty of these inputs.





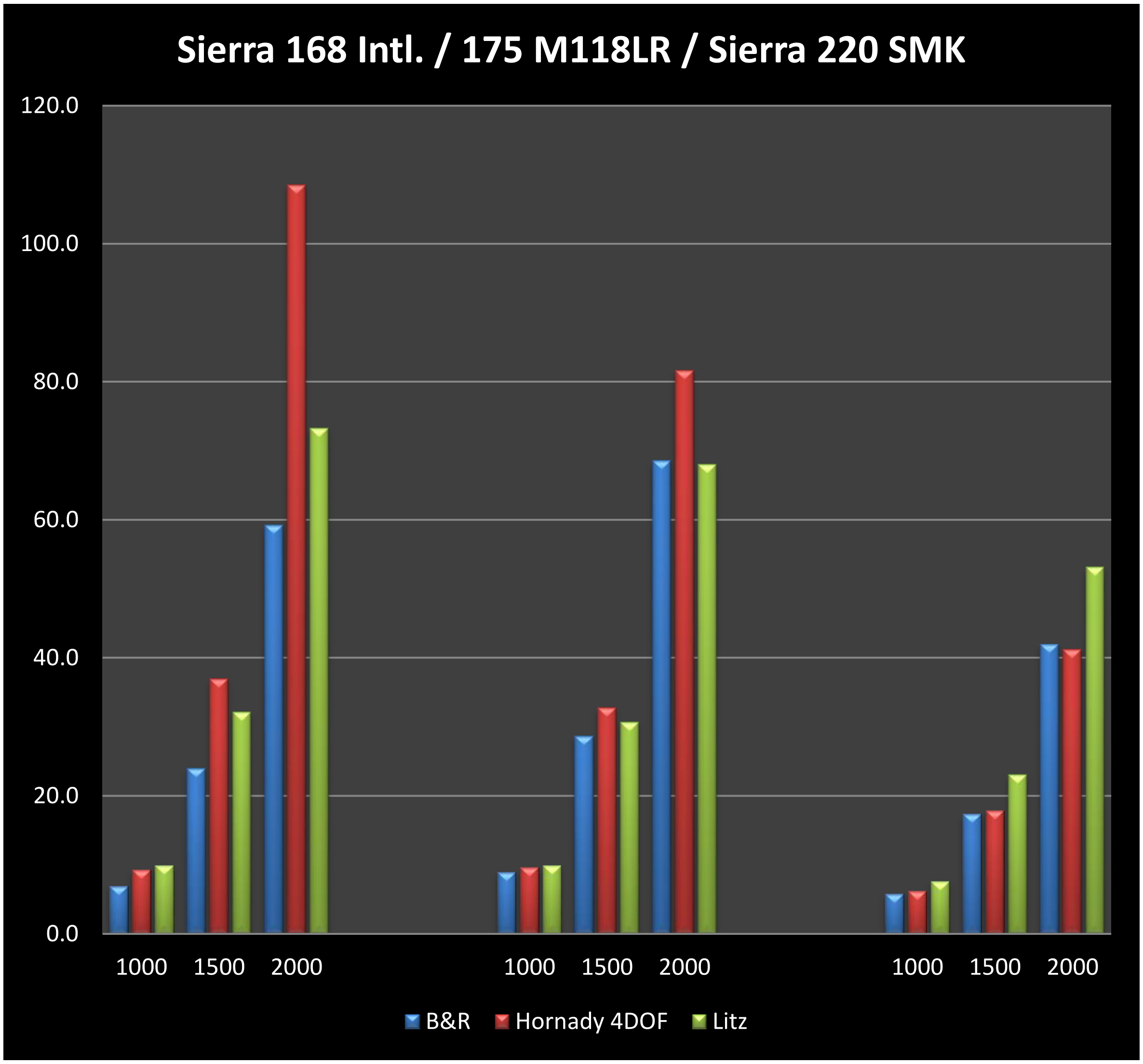
The slight decrease in **SD** with increasing **ToF** shown above for the B&R model is explained by not adjusting the velocity **V(t)** of the bullet as **ToF** is varied. The B&R model uses **V(t)** explicitly in many places.



In the 4-DoF case, the model response to a varying initial **Sg** with a fixed **ToF**, is quasi-linear and also exhibits a quite similar magnitude of the delta variation of **Sg** as the Litz and B&R models.

The 4-DoF (Modified Point-Mass, Lieske & Reiter, 1966) provides an estimate of the yaw of repose. This model considers the bullet rolling motion around its longitudinal axis of symmetry, called spinning motion. Therefore, this model presents four degrees of freedom: three translational coordinates for describing position and one for angular speed.

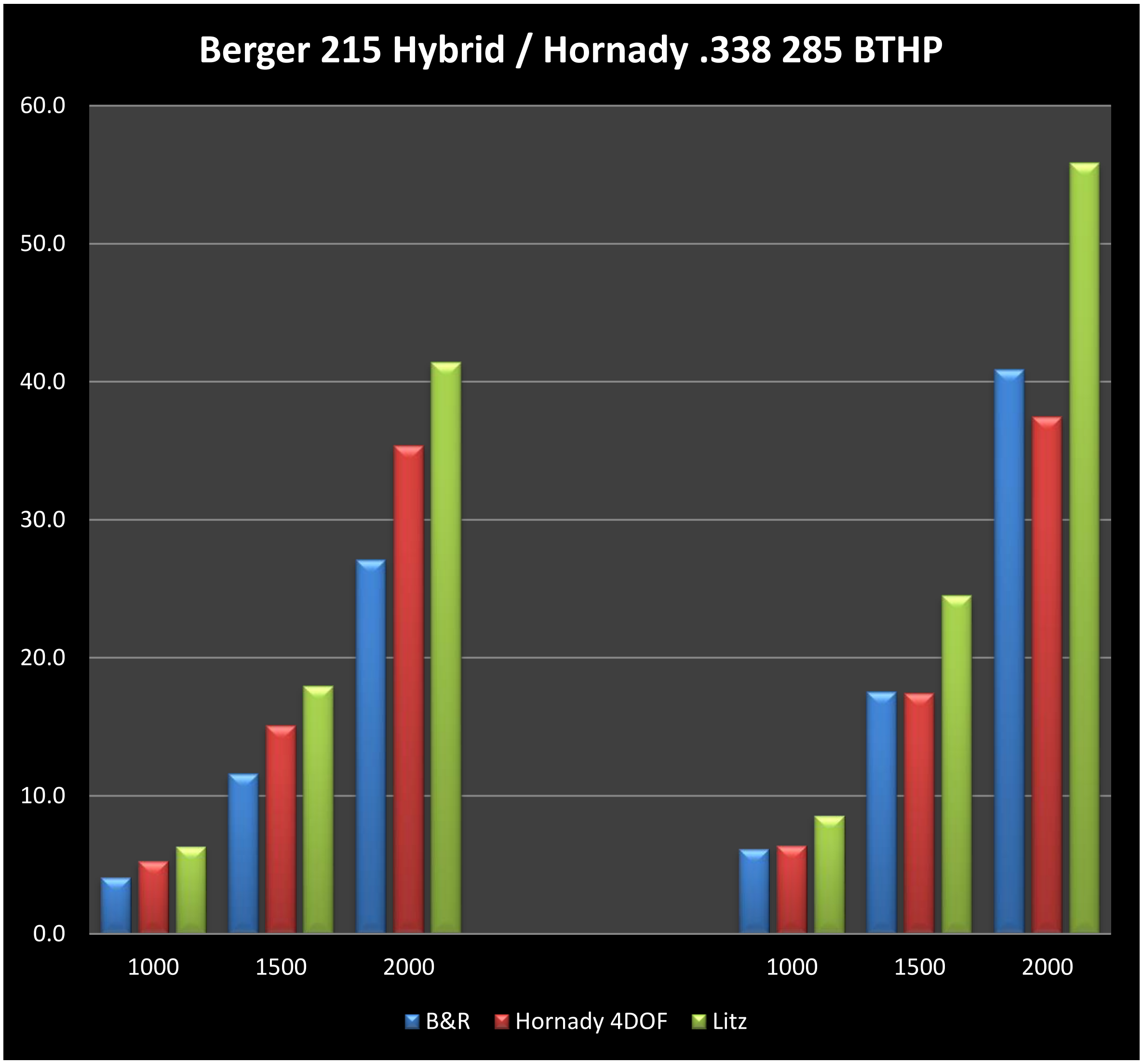
Some may argue that the underlying phenomena calls for a more elaborated multi-parameter analysis and while the concept is right, we chose to perform a single parameter analysis in order to compare to the Litz model which is a simple 2D model and as such does not relate the influence of one parameter over the other as the bullet goes down range, namely the aerodynamic coefficients.



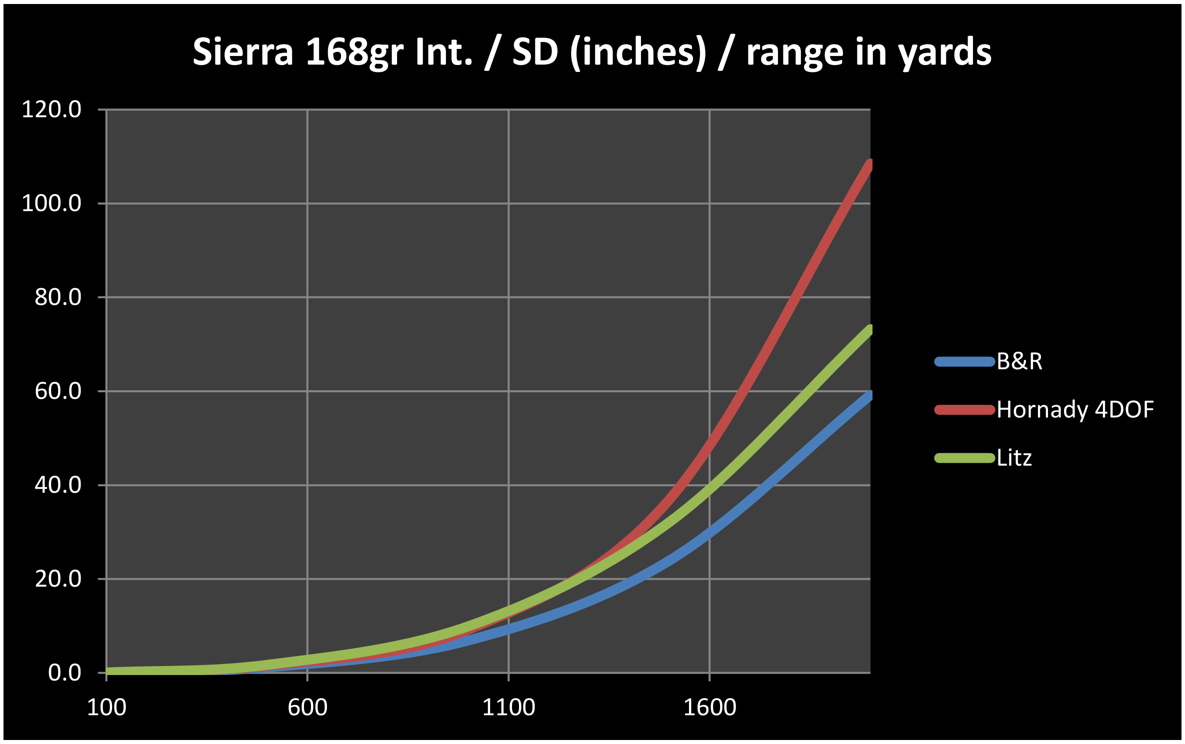
The Spin Drift is expressed inches, while each bullet is compared with the three different estimators, and grouped at 1000, 1500 and 2000 yards, which are typical ranges for extended Long Range shooting.

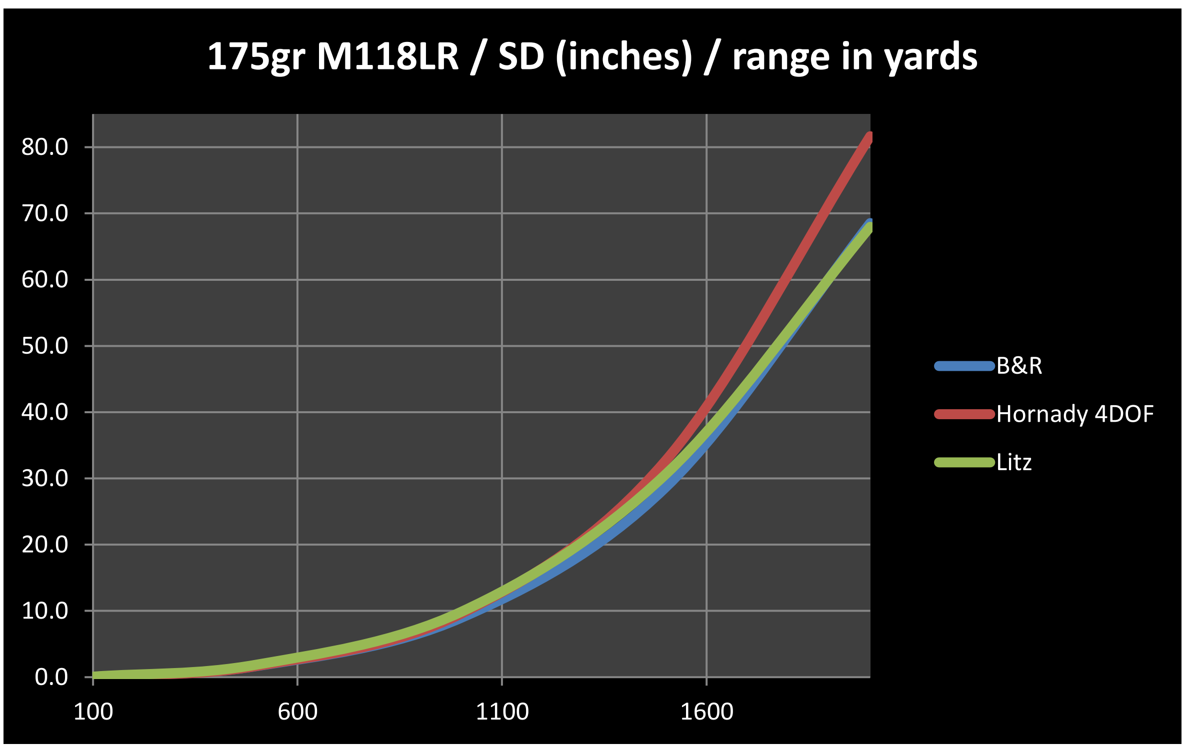
The Litz estimator does fair work, given its simple inputs, but its reliability is dictated by the underlying aerodynamics characteristics of the bullet, which are not accounted for in this simple linear approach.

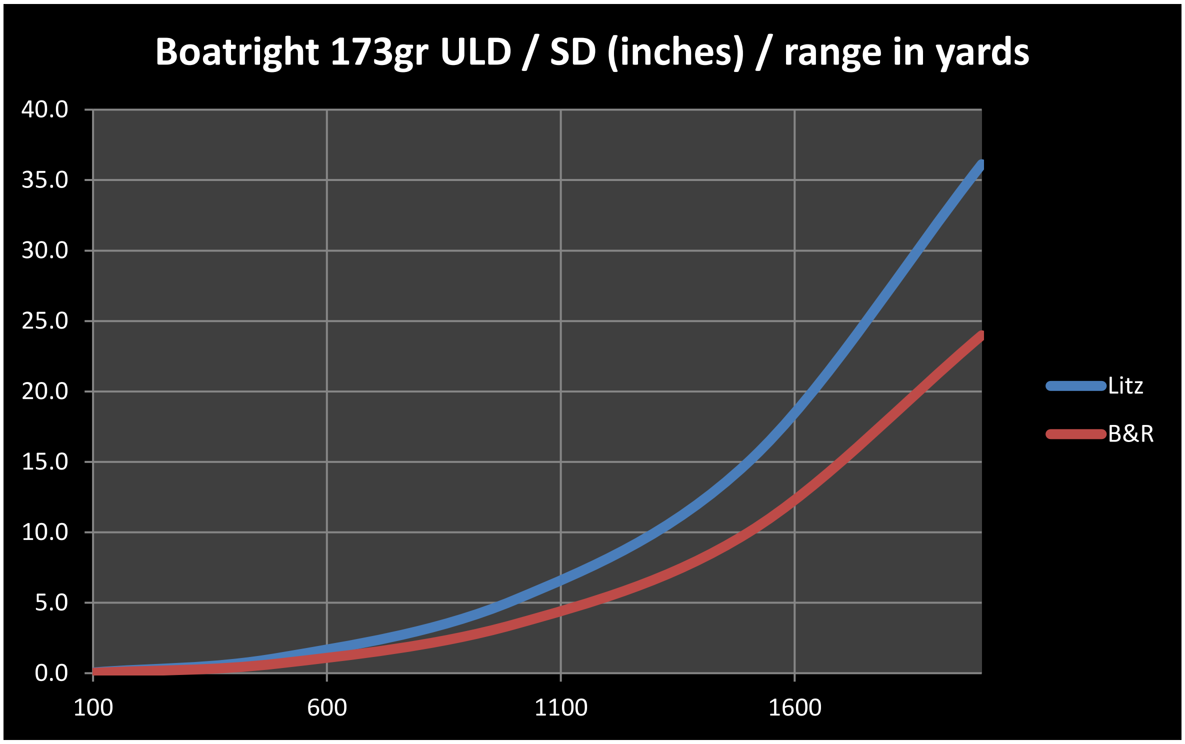
Consequently, as soon as the bullet does not exhibit certain properties that cannot be encompassed by **Sg** alone, its predictive accuracy is decidedly affected. In general terms, the Litz model tends to over predict **SD** in a significant way.

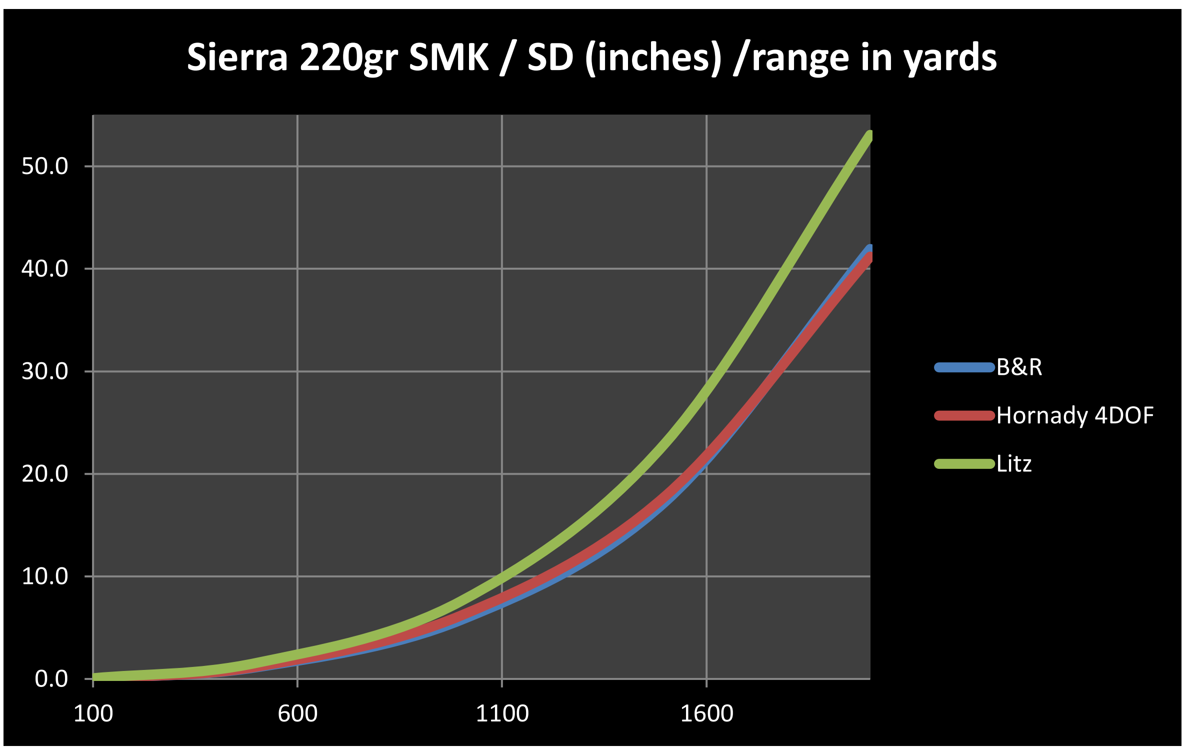


As can be appreciated, as the range increases, the difference among the estimators becomes larger. The practical side of this is that the correct method is of paramount importance when dealing with Extreme Long Range (ELR) shooting.


Sierra 168gr Int. / SD (inches) / range in yards
120.0
100.0
80.0
60.0
40.0
20.0
0.0
100
600
1100
1600
B&R
Hornady 4DOF
Litz


175gr M118LR / SD (inches) / range in yards
80.0
70.0
60.0
50.0
40.0
30.0
20.0
10.0
0.0
100
600
1100
1600
B&R
Hornady 4DOF
Litz


Boatright 173gr ULD / SD (inches) / range in yards
40.0
35.0
30.0
25.0
20.0
15.0
10.0
5.0
0.0
100
600
1100
1600
Litz
B&R


Sierra 220gr SMK / SD (inches) /range in yards
50.0
40.0
30.0
20.0
10.0
0.0
100
600
1100
1600
B&R
Hornady 4DOF
Litz

Closing Summary

Taken together, the implications of **Eq. 8** and **Eq. 19** determine the bullet and rifle characteristics which affect the size of the horizontal spin-drift **SD(t)** which will be seen in flat firing at a long-range target.

First, we see from **Eq. 8** that **SD(t)** displacement is always proportional to the bullet's **DROP(t)** in distance units from the projected axis of the bore at firing.

This implies that modern lighter-weight "flat shooting" bullets fired at higher muzzle velocities **V(0)** and retaining more velocity farther downrange (higher ballistic coefficient, lower drag bullets) will produce much less spin-drift **SD(t)** at any target distance compared to slower, higher-drag bullets. That is, **SD(t)** is roughly proportional to time-of-flight **t** to the target distance.

Second, according to **Eq. 19**, the size of the scale factor **ScF**, and thence the size of the spin-drift **SD(t)**, varies directly with the "potential ballistic drag force" $q(t)*S = \rho*V^2(t)*S/2$ in pounds. The ambient atmospheric density ρ varies with shooting conditions.

The rifle bullet's retained velocity **V(t)** depends upon its muzzle velocity **V(0)**, its mass **m**, and the integrated drag function CD_α of that bullet. The bullet's cross-sectional area $S = \pi*d^2/4$ varies with the square of the bullet's caliber **d**.

Third, the spin-drift **SD(t)** of the bullet is proportional to its yaw of repose angle $\beta_R(t)$ throughout its flight:

$$\beta_R(t) = (2\pi*g/t)\int[\omega_2(t)*V(t)]^{-1}\,dt$$

Both the coning rate $\omega_2(t)$ and the forward velocity **V(t)** of the bullet always gradually decrease, continually increasing $\beta_R(t)$ throughout the bullet's flight. The coning rate $\omega_2(t)$ is determined by the bullet's fixed inertial ratio **Iy/Ix** and by the remaining spin-rate $\omega(t)$ and slowly increasing gyroscopic stability **Sg** of the flying bullet.

The forward velocity **V(t)** of the flying bullet depends on its launch velocity **V(0)** and its coefficient of drag profile.

The yaw of repose attitude angle $\beta_R(t)$ is *increased* for bullets having larger numerical **Iy/Ix** ratios and higher initial stability **Sg**, but $\beta_R(t)$ is *decreased* by using faster twist-rate barrels and higher muzzle velocities **V(0)** to achieve that higher gyroscopic stability **Sg**.

Fourth, the spin-drift **SD(t)** is directly proportional to the small-yaw coefficient of lift $CL_\beta(t)$ of the bullet. Very-Low-Drag (VLD) and Ultra-Low-Drag (ULD) bullet designs usually have correspondingly reduced coefficient-of-lift functions at all supersonic airspeeds.

Fifth, and lastly, the spin-drift **SD(t)** of the bullet is inversely proportional to the weight **Wt** (or mass **m**) of that bullet.

All else being equal, bullets made with lower average material densities, such as turned brass or copper bullets, will weigh less and will suffer greater spin-drift.

These five **SD** effects combine multiplicatively in this analysis.

Some bullet and rifle design parameters recur in several of these different **SD** effects, and not always working in the same direction.

As modern long-range rifles and their bullets seem to be evolving toward lighter-weight, smaller-caliber, lower-drag bullets fired at higher muzzle velocities, these related incremental variations in design parameters combine algebraically to *reduce the spin-drift **SD** occurring on long-range targets*.

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