Calculating Aerodynamic Jump for Firing Point Conditions

-A novel and practical approach for computing the wind-induced jump perturbations-

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Introduction

This paper significantly updates and replaces the earlier published (2016) paper on this subject. A much improved estimator for the bullet's inertial ratio Iy/Ix is now incorporated.

The Coning Theory of Bullet Motions can be used in calculating aiming corrections in long-range shooting which go beyond those of the current 3 degree-of-freedom (3-DoF) point-mass trajectory propagations, but stop short of full 6-DoF trajectory simulations which compute the bullet's spin-rate and spin-axis pointing directions continually throughout its flight.

In particular, Coning Theory now allows **analytic calculation** of the very real and physical angular deflection of the trajectory which occurs when the just-fired rifle bullet first encounters the local atmosphere a few yards in front of the muzzle and is first exposed to the cross-track component of the horizontal wind (crosswind). Any prevailing crosswind at a shooting position near flat ground will likely be in purely horizontal laminar flow unless there is a significant obstruction nearby, upwind from the firing point.

We term this one-time angular deflection of the remaining trajectory the Crosswind Aerodynamic Jump (**CWAJ**). This analysis is an extension of the pioneering work in this field, done by ballistician Robert L. McCoy of the former Ballistics Research Laboratory (BRL) at the US Army's Aberdeen Proving Grounds.

This effect was first discovered by precision shooting benchrest and across-the-course riflemen who noted repeatable vertical wind effects while firing in purely horizontal crosswinds. It reminds us of the subtle perturbations first described by Dr. Franklin Weston Mann, during the early years of the twentieth century and published in his book “The Bullet's Flight from Powder to Target”.

Brought to the attention of Bob McCoy at BRL, he successfully formulated an analytical explanation for the vertical component of this **CWAJ** effect and demonstrated it using his own 6-DoF numerical flight simulator.

General form of the Aerodynamic Jump, as described by BRL.

$$J_a = K_y^2 \left(\frac{CL_a}{CM_a}\right)[iP\xi_0 - \xi_0']$$

CL_a = Aerodynamic Lift Force Coefficient

CM_a = Aerodynamic Overturning Moment Coefficient

$P = \frac{I_x}{I_y}\left(\frac{pd}{V}\right)$

ξ_0 = Initial Complex yaw angle

ξ_0' = Initial Complex yaw rate

J_a = tangent of the deflection angle due to jump

K_y = radius of gyration (in calibers) about a transverse axis through the CG of the projectile.

The regular small-arms trajectory modeling (fire solution predictions) accounts for the most prevalent characteristics of the trajectory and is most heavily influenced by the mass and drag characteristics of the bullet. It encompasses such effects as gravity drop and cross-wind drift. Both the Aerodynamic Jump and Lateral Throw-Off of the bullet produce angular deviations of the flight path, mainly due to launch disturbances and mass asymmetries within the bullet, respectively. Bias errors associated with these effects are normally removed through the rifle zeroing process.

However, these effects can also produce random errors that contribute to the ammunition dispersion and inconsistencies between the final fire solution and the actual Point of Impact.

As Murphy (1957) explains: “for an adequately stable bullet one of the most important causes of dispersion is the angle between the bore sight line and the" "effective" line of departure, i.e., “the jump”. (The effective line of departure is defined to be the line joining the muzzle and a distant point on a gravity free trajectory). This jump angle is determined by the impulse imparted by the gun barrel on launch (tip-off and muzzle whip), the momentum received during the blast regime (blast jump), and the influence of the aerodynamic force over the free flight trajectory (aerodynamic jump). In order to calculate blast jump and Aerodynamic Jump it is necessary to know the angular orientation of the bullet and the rate of change of orientation at the beginning of each regime. Results are therefore given in terms of the yaw and yawing rate at the beginning of blast and at the beginning of free flight"

In this study, we will focus only on the aerodynamic "jump" produced by the crosswind (normal component of the wind) at the firing site, having considered that the motion of a bullet can be separated into two general regions, namely the launch-disturbance region and the subsequent free-flight region.

In other words, the in-bore yaw and static unbalance yielding Lateral Throw-Off will not be considered, despite the fact that both perturbations when combined will account for the total Jump.

After traveling through the launch-disturbance region, the bullet enters free-flight. The motion of the bullet in free-flight is influenced by time-dependent side forces that cause the projectile **CG** to oscillate (swerve) about a mean swerve axis, as it travels to the target. As a result, the Aerodynamic Jump (which occurs in the free-flight region) due to the normal component of wind can be formulated as:

$$\mathrm{CWAJ} = -K_y^2 \left(\frac{CL_a}{CM_a}\right)\left(\frac{2\pi}{n}\right)\left(\frac{W_z}{V_0}\right)$$

$K_y^2 = \frac{I_y}{md^2}$ = Squared Transverse Radius of Gyration in calibers^2 (d^2)

n = twist rate in calibers per turn

V_0 = muzzle velocity

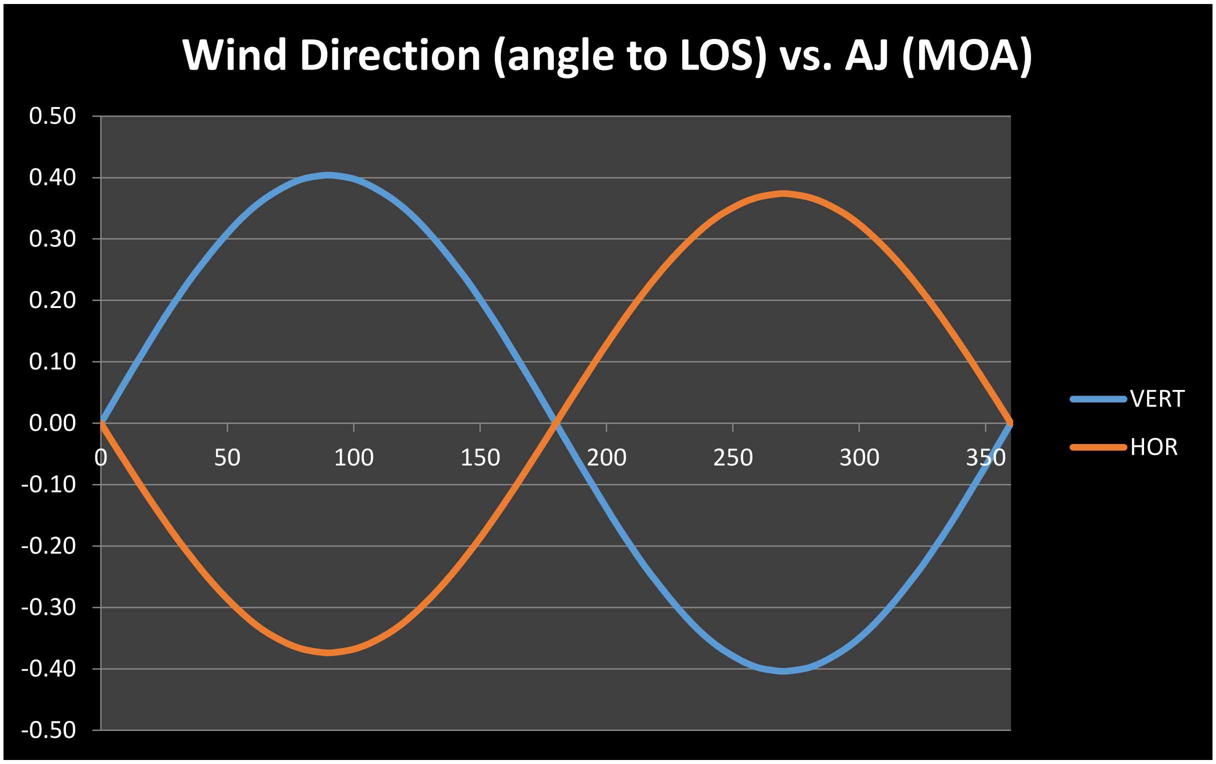
W_z = crosswind component of wind speed

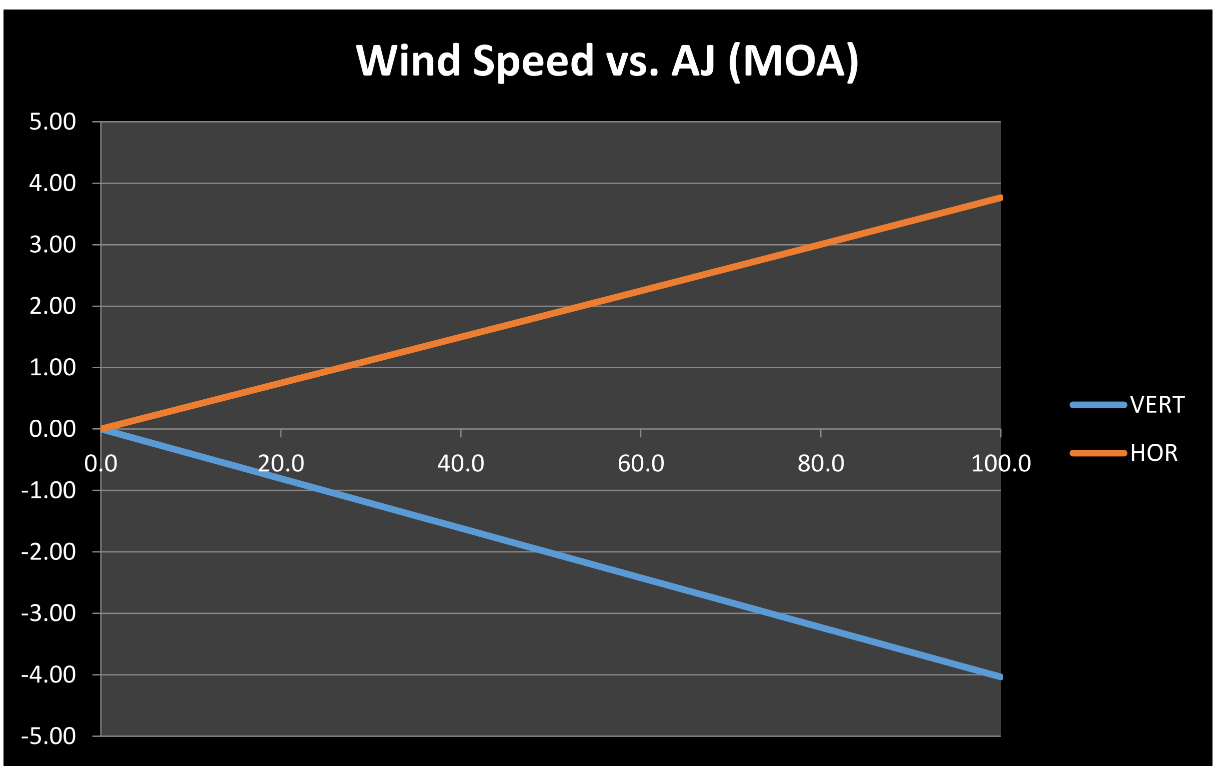
From now on, all references to the term **CWAJ** are directly related to the model developed and presented in this document, in order to avoid possible confusion with other methods.

From the above equation, it's clear, that the Aerodynamic Jump due to a crosswind, will always occur for any wind direction, except in the presence of a "pure range-wind" since its normal component becomes nil.

The following charts illustrate the effect of the Wind Direction and Speed on both, the Vertical and Horizontal deflections, due to the effect of the Aerodynamic Jump.

Signs will depend on the twist direction (left or right) as well as wind direction, in our case; the first graph corresponds to a right-twist.


Wind Direction (angle to LOS) vs. AJ (MOA)
0.50
0.40
0.30
0.20
0.10
0.00
-0.10
-0.20
-0.30
-0.40
-0.50
0
50
100
150
200
250
300
350
VERT
HOR


Wind Speed vs. AJ (MOA)
5.00
4.00
3.00
2.00
1.00
0.00
-1.00
-2.00
-3.00
-4.00
-5.00
0.0
20.0
40.0
60.0
80.0
100.0
VERT
HOR

Framework of the Analytical Solution

Having recently developed the analytical Coning Theory of Bullet Motions in flight, the core of this work, ballistician James A. Boatright is in a unique position to formulate its first practical application. Gustavo F. Ruiz of Patagonia Ballistics, LLC, worked out mathematical analysis along with numerous sensitivity analyses (derivatives) and has implemented a professional version of these CWAJ calculations for riflemen and supporting software which allowed us to test these algorithms over the wide range of variables which might be encountered in ballistic flight.

The utility of calculating these deflections is first, that they rely only upon data about the rifle, the bullet, and the current local shooting environment which can reasonably be measured either before-hand or in near real-time as the shot is about to be released.

Second, these deflections are formulated as final aiming corrections, making them suitable for use directly in the rifle's telescopic aiming device or sighting system.

In particular, this calculation of vertical-direction aerodynamic jump deflection angle uses only the crosswind information *at the firing point* which is routinely measurable and does not rely upon the more problematic estimation of wind effects farther downrange.

In fact, use of these **CWAJ** corrections does not affect either the traditional methods of estimating downrange wind effects or any new technology aiding in that effort. A portable computer having local environmental and wind sensor inputs can simply add the calculation of these **CWAJ** corrections to its repertoire of automatically calculated aiming corrections being made for each shot.

It's also essential to note, that neither the acceleration of gravity nor the Coriolis Effect play any role here since they are not aerodynamic forces, moreover their influences can simply be superimposed on the swerving motion of the bullet.

Aerodynamic Jump on First Encountering a Crosswind

Coning Theory shows that in flat-firing (super-elevation angle $\leq 5.7°$, or 100 milliradians), a spin-stabilized rifle bullet experiences an aerodynamic jump when it exits the muzzle blast cloud and first encounters the slowly moving undisturbed local air mass.

This **CWAJ** angular deflection can be resolved into its vertical and horizontal components which are calculated independently. While the existence of the vertical-direction CWAJ is now well known, we seek to discover if a similar horizontal-direction CWAJ might also occur.

These quantifiable angular deflections result from a one-time transient effect occurring during a brief time-span while the initial coning motion of the fired bullet is getting started. The **CWAJ** effect is completed within about 10 yards of the muzzle for any shoulder-fired rifle.

Thereafter, steady coning motion causes little or no further net angular deflection of the trajectory during the remainder of the bullet's flight. Because this particular **CWAJ** effect happens only once and occurs right in front of the muzzle, this calculated aiming correction is ***independent of time-of-flight*** and ***range to the target*** beyond the first 10 yards or so.

As stated before, implementation of this **CWAJ** deflections does not affect the overall problem of estimating down-range wind effects other than separately compensating for this one aspect previously compounded together with that problem.

The vertical-direction **CWAJ** reverses direction with the reversal of either crosswind direction or the sense of the rifling twist. The horizontal-direction **CWAJ** is always "downwind," so it always increases the size of the drag-induced (Didion) "wind drift" across the target regardless of the twist direction of the rifling.

Upon encountering a horizontal crosswind (actually the component of the prevailing horizontal wind normal to the firing azimuth), the first motion of the spin-axis of the bullet is always horizontally "downwind." In Coning Theory, this initial condition of horizontal first motion of the spin-axis fixes the relative amplitudes of the slow-mode and fast-mode epicyclic arms for a rifle bullet having a given initial gyroscopic stability factor **Sg**.

The initial aerodynamic lift-force acting horizontally through the center-of-pressure (CP) of the bullet not only produces an overturning moment initially rotating the nose of the bullet away from the approaching airstream, but it also initially begins moving the center-of-gravity (CG) of the bullet horizontally "downwind."

The gyroscopic response of the spinning bullet to this overturning moment rapidly begins to curve its spin-axis direction either upward or downward depending on its spin direction. This rotation of the curving lift-force vector begins the circular clockwise or counter-clockwise coning motion of the CG of the spinning bullet around its mean trajectory.

The transient vertical-direction **CWAJ** accumulates over the first 180 degrees of the epicyclic motion, in either rotational coning direction, before the rotating lift-force begins reversing its vertical direction.

However, since the transient horizontal **H-CWAJ** effect also starts in that same "downwind" horizontal direction, it can only accumulate over the first 90 degrees of epicyclic motion in either direction, depending on the sense of bullet rotation, before the rotating lift-force vector begins reversing its horizontal action on the CG.

We must limit the **H-** and **V-**direction integrations for evaluating these transient impulses at these two different "lift reversal" points.

The calculation of the **H-** and **V-**direction angular deflection components of the **CWAJ** are formulated differently because of this difference in their integration ranges.

Except for these transient start-up effects, the steadily rotating lift-force acting on the coning bullet (almost) directionally cancels during each successive coning cycle.

Riflemen are fortunate indeed that their spin-stabilized bullets always exhibit this coning motion causing rotational cancellation of the bullet's lift-force vector. Otherwise, a non-rotating lift-force acting always horizontally "downwind" on rifle bullets flying through a horizontal crosswind would produce at least an order of magnitude greater crosswind sensitivity at all ranges.

Gathering the Needed Data Values

Before running a full 6-DoF numerical simulation of a particular bullet's flight, we would have to make measurements and estimations of that bullet's many aeroballistics flight parameters and its mass properties. This process is difficult and expensive and has not been done for the vast majority of our long-range rifle bullets, even by their manufacturers.

However, we can estimate the aeroballistics parameters necessary for these **CWAJ** calculations from a few basic measurements of the particular bullet selected for our long-range shooting.

First, we need to specify the bullet's caliber **d** in inches; i.e., the groove diameter of the barrel from which it intended to be fired. All other bullet dimensions in inches will then be divided by this distance **d** to express those dimensions in the canonical units of "calibers."

We need the over-all length **L**, the nose length **LN**, the boat-tail length **LBT** (zero if none present), the diameter of its base **DB** (1.0 calibers for a flat-base bullet), and the diameter of the meplat **DM**, each expressed in calibers. We need the weight **Wt** of the bullet in grains.

We will also need the head-shape parameter **RT/R** for the ogive of this bullet, to avoid possible confusion we herein refer to this ratio as **RTR**. This parameter is the ratio of the generating radius **RT** of a tangent ogive at this nose length to the ogive generating radius **R** for this particular bullet.

We can calculate the exact radius **RT**, the curvature of a tangent ogive having this bullet's nose length **LN** and meplat diameter **DM**, but we still need to measure the radius **R** of the circular arc defining the nose-shape of this particular bullet design.

We need the ballistic coefficient (**BC**) of the bullet at its expected muzzle velocity $\mathbf{V_0}$. Depending on the standard drag function to which the BC is referenced, we need to interpolate the "zero-yaw" coefficient of drag $\mathbf{CD_0}$ for the reference projectile at our expected launch speed $\mathbf{V_0}$.

The input specified twist-rate **Tw** of our rifle barrel in inches per turn will be a positive value for right-hand twist barrels, and negative with left-hand twist rifling.

We need to know the initial gyroscopic stability **Sg** of this bullet fired in the prevailing ambient conditions from our rifle barrel having a given twist rate **Tw** in inches per turn. Ideally, **Sg** should be above **1.5** at launch.

$$\mathrm{Sg} = \left(\frac{I_x}{I_y}\right)\left(\frac{\varpi d}{V_0}\right)^2\left(\frac{2I_x}{\rho \pi d^5 CM_a}\right) > 1$$

Dr. Donald Miller's simplified (easy to get parameters) stability formulation for more-or-less "uniform-density" VLD-style rifle bullets can be used with the above bullet data to calculate the initial gyroscopic stability **Sg** for most long-range rifle bullets:

$$Sg = 30*Wt/[Tw^2*d*L*(1 + L^2)]$$

Caveat: Don Miller's method is for "muzzle only" calculations, it cannot be scaled for downrange decaying velocities and it is intended for supersonic-only Muzzle Velocity values. Outside these boundaries its accuracy will undergo a major loss.

Don's velocity and atmosphere correction factors, **fv** and **fa**, should also be used:

$$fv = [V_0/2800 \text{ feet/sec}]^{1/3}$$

$$fa = [(T + 459.67)/518.67]*[29.92/P]$$

Where,

T = Ambient air temperature in Fahrenheit degrees at firing point

P = Absolute barometric air pressure in inches of mercury (inHg)

Then,

$Sg \leftarrow Sg*fv*fa$ (adjusting the previously calculated Sg value).

A more detailed version of this stability formulation was developed by Dr. Michael Courtney in his paper "A Stability Formula for Plastic-Tipped Bullets" and is available for estimating the **Sg** of bullets having significant hollow cavities or low-density polymer tips. In particular, the $\mathbf{L^2}$ term in the denominator is often replaced with the square of the "metal length" (in caliber units) for plastic-tipped bullets.

From measurements of the environmental conditions and wind ***at the firing point***, we need the air density **rho** and the "speed of sound" **a** (Mach 1.0 in these ambient conditions) in suitable units. We also need to know the horizontal wind speed and direction at the firing point at the time of the shot.

We need the current local wind speed **Vw** in feet per second if that is how muzzle velocity $\mathbf{V_0}$ is given. The direction *from which* the wind is blowing toward the firing point is entered in horizontal "clock numbers" relative to the firing direction being at 12:00 o'clock.

For example, a near-surface wind blowing at the firing point from 9:00 o'clock is a full value left-to-right crosswind, with the **CWAJ** initially deflecting the trajectory <u>rightward</u> and <u>downward</u> (for a bullet launched with a right-handed spin). For a bullet fired from a left-hand twist barrel in this same crosswind at the firing point, the **CWAJ** deflections would be <u>rightward</u> and <u>upward</u>.

Calculating the Initial Spin-Axis Motion Parameters

We find the initial Mach-speed **M** of the bullet by dividing the expected muzzle velocity of the bullet (**V_0**) by the calculated "speed of sound" (**a**) in the atmosphere prevailing at the firing point. The **Mach 1.0** airspeed **a** is **1116.45 feet per second** for the ICAO-standard sea-level atmosphere used here. Both speeds should be given in the same units. As an adiabatic acoustic compression-wave, the speed of sound propagation through the air (**a**) varies primarily with air temperature and secondarily with the elasticity, density, viscosity, and make-up of the mixture of gasses called "air."

We can refer to the drag curve shown below in tabular form for the *G7 Reference Projectile* used in establishing the **BC(G7)** value for this bullet to interpolate the reference coefficient of drag **CDref7** for that G7 projectile at its (assumed supersonic) Mach-speed **M** muzzle velocity.

Mach	1.20	1.30	1.40	1.50	1.60	1.80	2.00	2.20	2.50	3.00	3.50	4.00
CDref7	0.388	0.373	0.358	0.344	0.332	0.312	0.298	0.286	0.270	0.242	0.215	0.194
CDref1	0.639	0.659	0.663	0.657	0.647	0.621	0.593	0.569	0.540	0.513	0.504	0.501
Ratio	1.647	1.767	1.852	1.910	1.949	1.990	1.990	1.990	2.000	2.120	2.344	2.582

These tabulated values for the dimensionless reference drag coefficients **CDref7** of the G7 projectile at different supersonic Mach-speeds are said to be the "zero-yaw" drag data for that G7 reference projectile flying exactly nose-forward through the air. The supersonic drag curve for the G1 Reference Projectile is also shown in this table. The ratio of **CDref1/CDref7** shows the conversion factor to relate the Ballistic Coefficients, **BC(G1)** and **BC(G7)** at each Mach-speed.

When a rifle bullet flies through the air with a significantly non-zero angle-of-attack (i.e., yaw angle), its effective coefficient of drag **CD** will be increased significantly. Since real rifle bullets are statically unstable, they must be dynamically spin-stabilized and thus will always experience coning motion during flight. The coning angle is identically the aerodynamic angle-of-attack of the flying bullet.

Ballistic Coefficient values determined from outdoor firing-test data typically correspond to this "yaw drag" augmented total drag coefficient for an angle-of-attack equal to that tested bullet's unspecified, but likely significant, coning angle. On the other hand, a bullet's advertised **BC** values might tend to be a little optimistically high, implying an equally optimistic (lower) coefficient of drag **CD**.

With those caveats about air-drag coefficients in mind, we can back-figure an estimate of the "zero-yaw" coefficient of drag **CD_0** for our bullet at its launch speed **V** from its **BC(G7)** value, for example, as:

$CD_0 = (CDref7)*[Wt/(7000*d^2)]/BC(G7)$

We then estimate the coefficient of lift **CL** for our subject bullet at Mach-speed **M** corresponding to our expected muzzle velocity V_0 by the following procedure:

$RT = \{LN^2 + [(1 – DM)/2]^2\}/(1 – DM)$

$LFT = SQRT(RT – 0.25)$

$LFC = LN/(1 – DM)$

$LFN = LFT*RTR + LFC*(1 – RTR)$

$B = SQRT(M^2 – 1)$

$CL = 1.974 + 0.921*(B/LFN)$

RT is the generating radius for a tangent ogive of length **LN** after being truncated by a meplat of diameter **DM**. **LFT** and **LFC** are the calculated full ogive lengths (carried to a sharp point) for tangent and conical ogives, respectively, having these same values of **LN** and **DM**. These calculations are exact geometric relationships for any bullet having a truncated circular-arc head-shape.

The bi-linear interpolation for **LFN**, the calculated full length of this particular bullet's nose, is as accurate as the **RT/R** value upon which it is based. **RT/R** varies from **0.00** for a purely conical nose, through **0.50** for a lowest-drag secant ogive nose, to **1.00** for a tangent ogive nose. The value **B** is a Mach-dependent auxiliary parameter frequently used in aeroballistics. This estimate of the lift coefficient **CL** for this bullet at this Mach-speed **M** should be accurate to well within 10 percent (1 sigma) for any long-range bullet likely to be used.

This calculation (extracted from McCoy's Interim Lift estimating program, **INTLIFT**) assumes the Mach-speed **M** of the fired bullet will normally exceed **Mach 2.0** at the muzzle of the rifle. We are estimating the coefficient of lift, and consequently the aerodynamic lift-force experienced by this bullet at any reasonable angle-of-attack, based solely upon its complete ogive length **LFN**.

Long-nosed, very low drag (VLD) bullets will generate less aerodynamic lift-force, as well as less drag-force, at each Mach-speed and angle-of-attack. If the bullet design also incorporates an aerodynamically effective boat-tail (as with a VLD bullet), that bullet will generate slightly less lift-force than this estimate. [However, we are not estimating that small lift correction here.]

We can estimate the ratio of the bullet's second moment of inertia **Iy** about a transverse axis through its center of gravity (CG) to its second moment of inertia **Ix** about its spin-axis from:

$Iy/Ix = f_{1A}(LL,h)/[30*(1 – 4*h/5)]$

where

Wt = Bullet Weight (in grains)

Wtcalc = $(\pi/4)*\rho_p*d^3*LL*(1 - 2*h/3)$ (in grains)

ρ_p = Average Density of Projectile (in grains/cubic inch)

ρ = Ambient Air Density (in grains/cubic inch)

$\rho = 4.051*$(Air Density in pounds per square foot)

LL = L - LN +LFN (in calibers)

h = LFN/LL

$f_1(LL,h) = 15 - 12*h + LL^2*(60 - 160*h + 180*h^2 - 96*h^3 + 19*h^4)/(3 - 2*h)$

$f_{1A}(LL,h) = (Wt/Wtcalc)^{0.894}*f_1(LL,h)$

$f_2(h) = (18 - 24*h + 7*h^2)/(18 - 12*h)$

$f_{2A}(h) = (2.62/3.41)*f_2(h)$.

For a monolithic C145 or C147 copper ULD bullet, **ρ_p = 2255.8 gr/in^3**. For a thin-jacketed, pure lead-cored match rifle bullet having no appreciable hollow cavities, **ρ_p = 2750 gr/in^3**. For a thick-jacketed, lead-alloy-cored military rifle bullet, **ρ_p = 2600 gr/in^3**. For a monolithic bullet made of C360 brass, **ρ_p = 2120 gr/in^3**.

This inertial ratio **Iy/Ix** should range from about **7.0** to **15.0** for long-range rifle bullets depending upon the bullet's radial weight distribution over its length. This estimation procedure assumes a typical long-range rifle bullet shape and a uniform material density without hollow cavities. The polynomial **$f_1(LL,h)$** in **LL** and **h** is derived for a cone-on-cylinder solid model of the rifle bullet as used by Ing. Dr. Beat P. Kneubuehl of Thun, Switzerland, in his 1989 paper, *"What is the Maximum Length of a Spin-Stabilized Projectile?"*

Another method of estimating the initial gyroscopic stability **Sg** can be used in absence of a reliable input value. This alternative estimator is also based on the work of Dr. Kneubuehl.

$Sg = 0.2339*(\rho_p/\rho)*[(5 - 4*h)*Tan(\pi/n)]^2/[f_{1A}(L,h)*f_{2A}(h)]$

where **n** is just the twist-rate of the rifling given in calibers **d** per turn.

We find the initial spin-rate **p** (in revolutions per second, or as a frequency in hertz) of the bullet right out of the muzzle from:

$p = (12\ \text{inches/foot})*[V_0/ABS(Tw)]$

To find the initial rates (in hertz) of the gyroscopic precession **f_2** and nutation **f_1** motions of the spin-axis of this bullet right after launch, we should first realize that gyroscopic stability **Sg** can always be expressed as:

$Sg = (R + 1)^2/(4*R)$

Where this stability ratio **R** is the ratio of the inertial gyroscopic nutation and precession rates **(f_1/f_2)**, and is unrelated to the ogive generating radius **R** mentioned earlier. The two stability factors, **Sg** and **R**, must each be greater than **1.00** for technical gyroscopic stability, and **Sg** must be greater than **1.250 (R = 3.0)** for this formulation to work properly. Ideally, the initial value of **Sg** will exceed **1.50**, and **R** will exceed **3.732** at launch. The stability ratio **R** acts as a more sensitive indicator of gyroscopic stability than does **Sg** itself.

We can invert the equation above to find the initial value of this gyroscopic rate ratio **R** by solving for the correct root of that quadratic equation since we have input or estimated an initial value for **Sg** in launch conditions:

$R = 2*\{Sg + SQRT[Sg*(Sg - 1)]\} - 1$

From Tri-Cyclic Theory, we know that at any time during the bullet's flight the sum of these two inertial gyroscopic rates ($f_1 + f_2$) is always given by the bullet's instantaneous spin-rate **p** divided by its (fixed) second moment ratio **Iy/Ix**:

$f_1 + f_2 = f_2*(R + 1) = p/(Iy/Ix)$

$f_2 = (f_1 + f_2)/(R + 1)$

$f_1 = R*f_2$

Having found the bullet's initial gyroscopic precession f_2 and nutation f_1 inertial rates, we are now able to calculate the *epicyclic motion* of its spin-axis during the brief time interval $n*T_N$, the period in seconds of the first **n** relative nutation (fast-mode) cycles of the bullet's spin-axis. There are **R – 1** relative f_1 fast-mode cycles per slow-mode coning cycle occurring at the f_2 gyroscopic precession rate. This integer **n** is the number of complete initial fast-mode cycles needed to span the period during which the coning motion is getting started, and **n** is found from:

$n = 1 + INT[(R - 1)/4]$

The period T_N of this first relative nutation cycle is:

$T_N = 1/(f_1 - f_2) = 1/[f_2*(R - 1)]$

We will calculate the time-average of the epicyclic pitch attitude of the spin-axis (including angular vector cancellation effects) during these first **n** fast-mode cycles from time **t = 0** at the bullet's first encounter with the ambient atmosphere to time $t = n*T_N$ when the n^{th} inward-pointing cusp of the epicyclic motion occurs.

We will then use this average pitch attitude angle as a vertical-direction aerodynamic angle-of-attack in the coning force **Fc** expression from Coning Theory to evaluate how these transient forces shift the CG of the bullet off-track vertically and, thus, permanently deflect the remaining trajectory of the bullet by a vertical angular amount. The coning force **Fc** is essentially linear with angle-of-attack for small coning angles. All angular arguments are calculated in units of radians.

The magnitude of the centripetal coning force **{F_C}** for an angle-of-attack **α** is given from Coning Theory as:

{F_C} = q*S*Sin(α)*[CLα+CD]

The initial magnitude **GAMMA** of the horizontal crosswind angle-of-attack experienced by the bullet as it first enters the local air-mass, undisturbed by firing, is found from:

GAMMA = {Vw*Sin[(Pi/6)*(clock angle)]}/V

Over the time interval **n*T_N** of the first **n** relative nutation cycles, the average vertical direction angle-of-attack **PITCH** is found from Coning Theory as:

PITCH = GAMMA*[(R^2 – 1)/(n*2*Pi*R)]*{1 – Cos[n*2*Pi/(R – 1)]}

This averaged vertical angle-of-attack is usually just a few milliradians in magnitude.

In this model, we are calculating a single effective average pitch attitude of the spin-axis of the bullet over the first **n** (complete) relative nutation cycles.

The value of **n** is calculated so that we are averaging over at least **Pi/2** radians, but not more than **Pi** radians, of the epicyclic motion of the bullet's spin-axis about the direction from which the wind is approaching the flying bullet.

The first requirement (**> Pi/2** radians) has been found necessary to handle very high-stability (**Sg > 3.0**) cases where the epicyclic rate-ratio **R > 10.0** and the long-nosed, low-lift bullet has a very rigid spin-axis.

The second requirement (**<Pi** radians) is necessary to require that marginally-stable (**Sg < 1.563; R < 4.0**) and stable bullets (**Sg < 1.800; R < 5.0**) complete their first nutation cycle before the coning force **Fc** reverses its vertical direction.

At the lower limit of bullet stability (**Sg = 1.250: R = 3.0**) used here, the first relative nutation cycle occupies the entire first **Pi** radians (180 degrees) of the slow-mode, **f_2**-rate, coning motion of the bullet's spin-axis.

Calculating the Yaw-Drag Coefficient

First we estimate the side-view, cross-sectional area *aspect ratio* **AR** in calibers for this bullet:

AR = L – (2/3)*[LFN + LBT*(1 - DB)]

The term **(2/3)*LFN** is an approximation for the distance from the nose tip to the centroid of the cross-sectional area for a typical bullet's full-length ogive, and the length adjustment for a conical boat-tail is geometrically accurate. We are shortening the actual bullet length **L** by those two adjustments to estimate an equivalent rectangular side-view length metric **AR** in calibers for this bullet.

Note that the boat-tail adjustment goes to **zero** either if **LBT = 0.00** or if **DB =1.00**, and the adjustment goes to 2/3 of the boat-tail length **LBT** for a full-length tail-cone tapering to a point (**DB = 0.00**). The aspect ratio **AR** tells us how much side area of the bullet is exposed to the airstream as the bullet yaws in flight.

Then, the secondary yaw-drag coefficient **CDa** can be found from:

CDa = 1.33*(1.41 – 0.18*M)*[9.825 – 3.95*M + (0.1458*M – 0.1594)*CL^2*AR]

This expression comes from McCoy's INTLIFT program for Mach-speed **M > 2.0**. Note the heavy squared dependence on the coefficient of lift **CL**, which is as it should be for yaw-drag. These yaw-drag estimates are accurate within 25 percent (1 sigma).

The yaw-adjusted coefficient of drag **CD** is then calculated from our previous zero-yaw coefficient **CD0** for our bullet's coning angle **GAMMA*R/(R – 1)** as:

CD = CD0 + Sin^2[GAMMA*R/(R – 1)]*CDa

Because of the **Sin^2[yaw angle]** multiplier in this aeroballistics formula, the adjusted coefficient of drag **CD** remains an even function in yaw angle, and the yaw-drag adjustment is quite small until the coning angle-of-attack exceeds about **5.7 degrees (0.10 radians)**, which can occur at times.

We are now able to calculate the expected vertical deflection angle of the trajectory attributable to this averaged transient vertical angle-of-attack, termed **PITCH** angle (in radians).

Calculation of CWAJ Deflections & Aiming Corrections

First, we need to calculate the "dynamic pressure" **q** in units of pounds per square inch:

$$q = (rho/2)*V_0^2$$

Then we need to find the cross-sectional area **S** in square inches of the bullet as viewed from ahead:

$$S = (Pi/4)*d^2$$

We can now calculate the vertical and horizontal direction cross-track ***impulses*** imparted to the CG of the rifle bullet, **Jv** and **Jh**, respectively, in impulse units of pound-seconds (average force multiplied by time duration). The vertical-direction impulse **Jv** is found by evaluating the average coning force **{Fc}** in the pitch-direction over the time interval $n*T_N$:

$$Jv = SIGN(Tw)*(n*T_N)*(q*S)*(CL + CD)*Sin(PITCH)$$

The product of the two odd mathematical functions, **SIGN(Tw)** and **Sin(Pitch)**, keeps the directional sense of the vertical-direction cross-track impulse **Jv** correct in all cases.

[An odd function **f(x)** is one in which **f (-x) = -f(x)** for all values of **x**. For all real values of **x**, **SIGN(x)*ABS(x) = x**, so **SIGN(Tw)** is an odd function, as is the trigonometric sine-function.]

The horizontal-direction impulse **Jh** is always in the "downwind" direction for either sense of bullet spin; i.e., first motion of the nose of the bullet is horizontally "downwind" followed by the epicyclic motion curling either upward or downward depending upon bullet rotation sense. For this type of coning motion, the initial condition of *horizontal first motion* importantly determines the relative magnitudes (angular lengths) of the slow-mode and fast-mode arms of the epicyclic motion of the bullet's spin-axis orientation.

We must use a different approach in evaluating the horizontal-direction impulse **Jh** because the coning force **Fc** reverses its horizontal direction after only **Pi/2 radians** (90 degrees) of epicyclic motion.

This formulation looks primarily at the slow-mode precession of the bullet's spin-axis, and incorporates the (smaller) fast-mode nutation motion only as an average increase in the coning angle **GAMMA**.

We have to stop integrating the horizontal impulse **Jh** after only the first **Pi/2 radians** (90 degrees) of coning motion minus $Sin^{-1}(1/R)$, at time $(T_2/4)*(2/Pi)*[Cos^{-1}(1/R)]$, by which time the horizontal lift-force begins reversing, and the transient horizontal **CWAJ** is completed.

Recall that:

$$T_2 = 1/f_2 \text{ (in seconds)}$$

Then, from Coning Theory, the horizontal impulse **Jh** can be calculated to be:

Jh = - $(T_2/4)*(2/Pi)^2$ *GAMMA*SQRT [(R + 1)/(R – 1)]*(q*S)*(CL + CD)

This formulation incorporates an extra factor of

(2/Pi)*[SQRT(R^2 -1)]/[R*Cos^{-1}(1/R)]

to account for averaging the directional cancelling of the **Cosine(coning phase angle)** function over most of its first quadrant: **Pi/2 radians - Sin^{-1}(1/R)**, or **Cos^{-1}(1/R)**. Fortunately, the **R**-expressions simplify and the two inverse cosine factors divide out for calculating the horizontal impulse **Jh**.

To evaluate the cross-track deflection angles caused by these cross-track impulses, we first need to calculate the magnitude **MOM** of the bullet's linear momentum vector at this beginning point of its flight in units of pound-seconds:

MOM = m*V_0 = [Wt/(7000*g)]*V_0

Where **m** is the mass of the bullet and the nominal acceleration of gravity **g** is taken to be:

g = 32.174 feet/$second^2$

Finally, the **CWAJ** vertical and horizontal deflections, **in milliradians (mils)**, are:

V-CWAJ = 1000*Jv/MOM

H-CWAJ = 1000*Jh/MOM

To convert **V-CWAJ (vertical)** and **H-CWAJ (horizontal)** to actual aiming corrections, **EL** (elevation) and **WI** (windage), do the following:

EL = - (V-CWAJ)

WI = - (H-CWAJ)

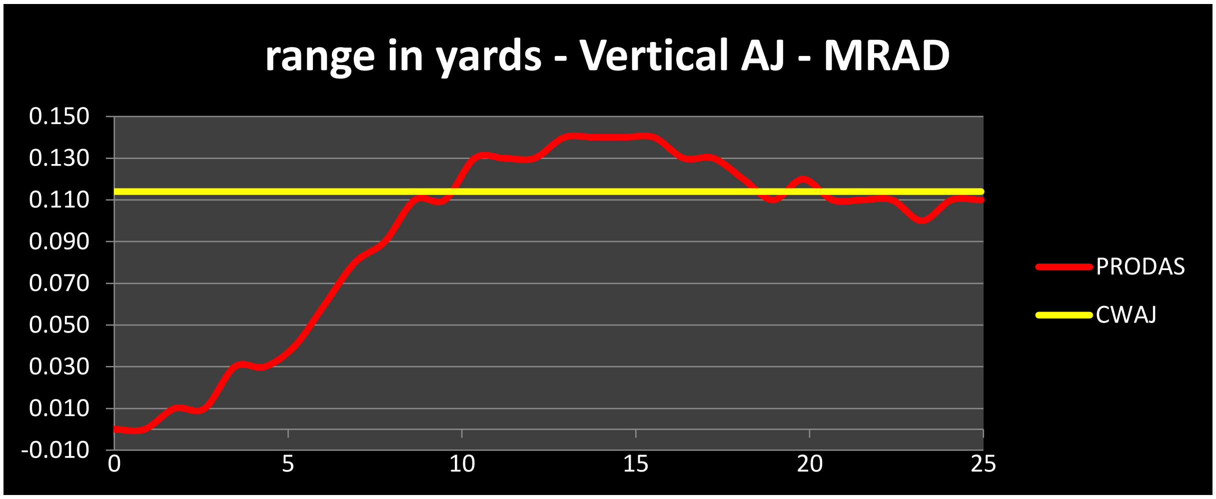
The algebraic signs have been reversed for these aiming adjustments to agree with rifle sight adjustment conventions: "**EL** positive" means adjusting the bullet strike upward on the target, and "**WI** positive" means moving the impact point rightward on the target.

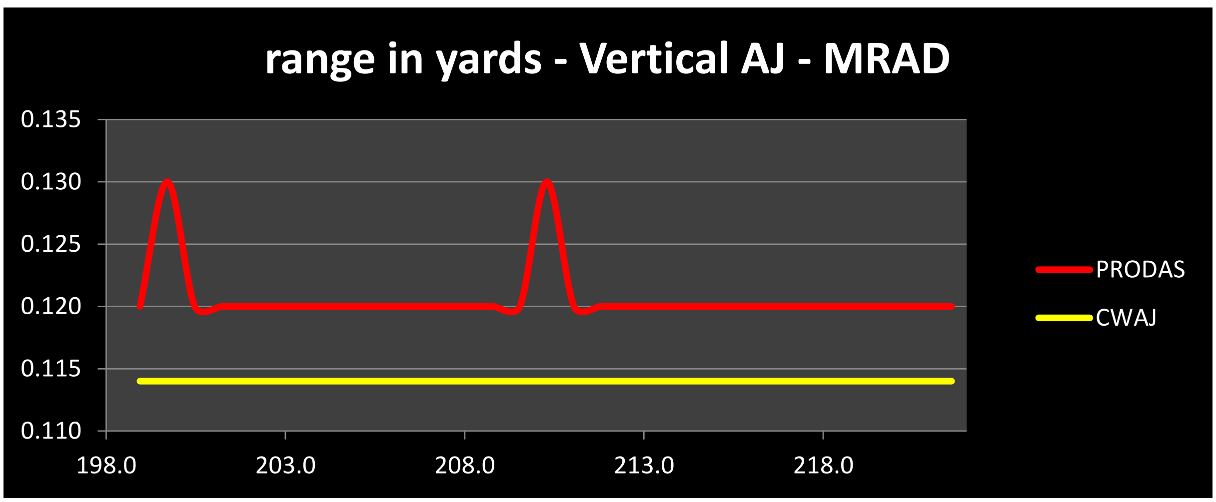
These values may be multiplied by **3.4377 MOA/milliradian** to convert these aiming corrections into minutes-of-angle (**MOA**).

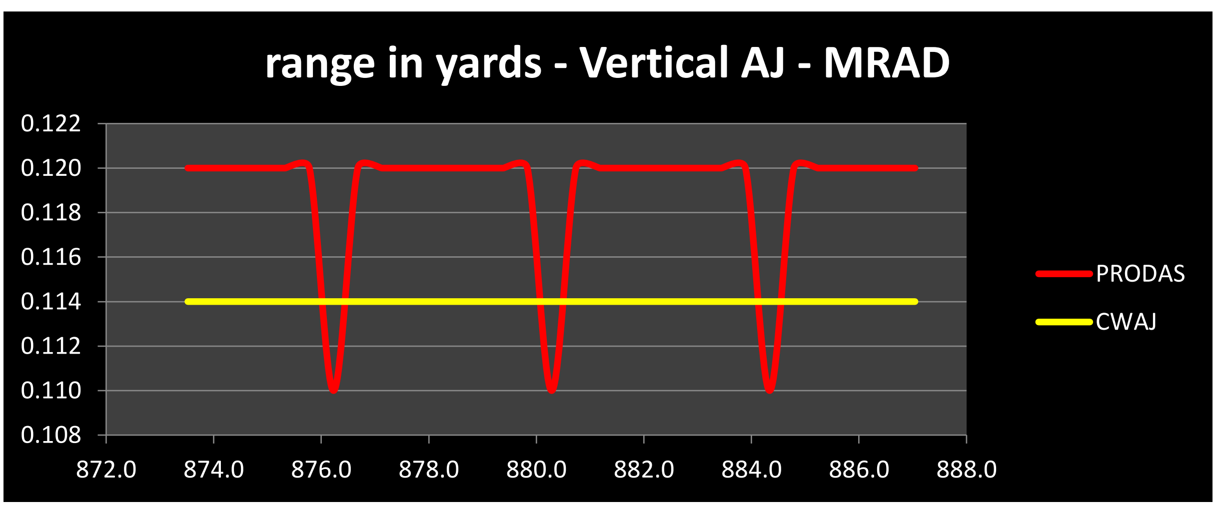
This vertical-direction **Elevation Aiming Correction (EL)** offsets a permanent one-time angular deflection of the trajectory which is independent of *time-of-flight* and *range to the target*, and so it is a very important aiming correction for riflemen to use in all outdoor shooting.

PRODAS 6-DoF simulator "bullet DROP" data streams show that the **CG** of the bullet follows this vertically deflected trajectory to all ranges beyond 20 yards.

The **V-CWAJ** calculations agree with the PRODAS data (M118 Mk316 Mod 0) within a few percent at all ranges.







The PRODAS V-CWAJ data are derived by differencing the **DROP** values of the CG from the bore axis, point by point, for two back-to-back PRODAS runs differing only in the constant crosswind value: 10 MPH constant Left-to-Right crosswind versus No-Wind, each with the same right-hand rifling twist.

Unfortunately, the **DROP** data in milliradians about the firing point are reported only to two decimal places in these PRODAS runs. Our single calculated **V-CWAJ** value for this bullet fired in this 10 MPH crosswind is **0.114 milliradians.**

The PRODAS V-CWAJ value accumulates to match our 0.114 mrad value at just less than 10 yards downrange (after 180 degrees of initial coning rotation for this bullet), over-shoots slightly, and then steadies out at very nearly our calculated value, even to maximum supersonic range for this simulated firing.

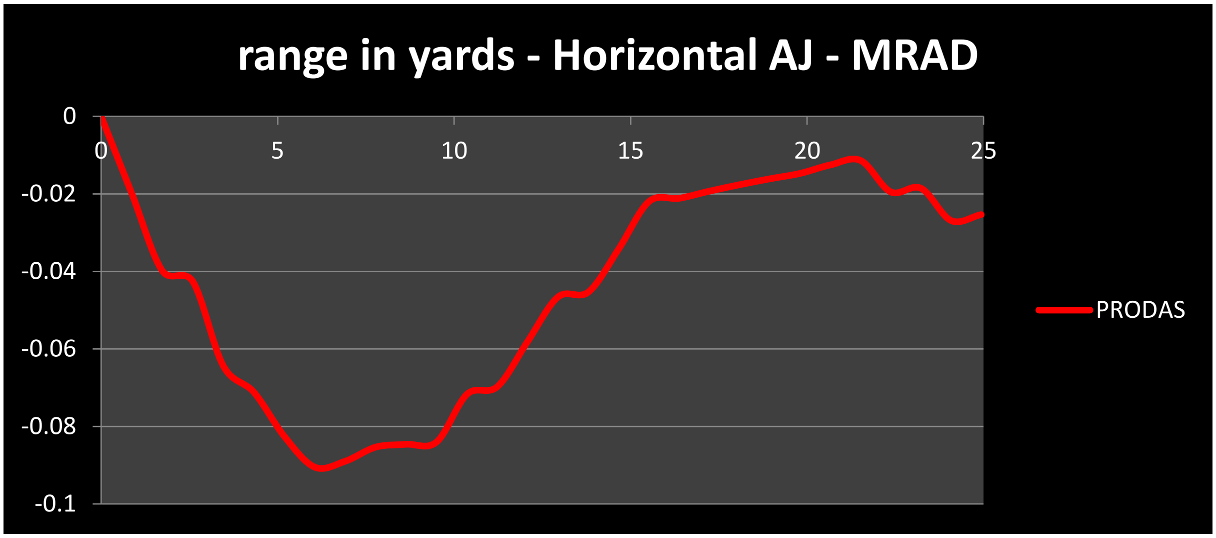
However, this calculated **Windage Correction (WI)** only applies at about 10-yards distance from the muzzle and is nearly **zero** after the first 20 yards of flight.

Thereafter, the size of the **horizontal-direction angular deflection** quickly decreases exponentially with time-of-flight in PRODAS simulation runs, trending strongly toward **zero correction** at long ranges, as the coning bullet subsequently cancels-out this initial **H-CWAJ** deflection.

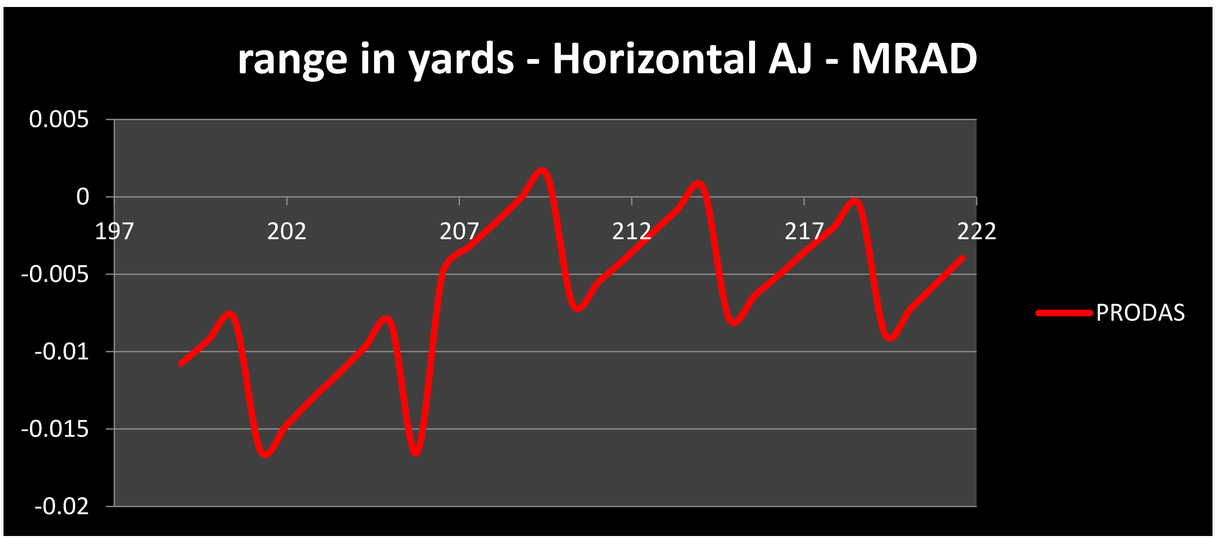
This PRODAS H-CWAJ data is derived by differencing the horizontal **DRIFT** of the bullet's CG in milliradians for the same two runs mentioned above (to remove the "spin-drift") and then subtracting out, point-by-point an accurate calculation of the Didion wind drift (based on drag-induced flight delay), leaving only the H-CWAJ values.

Even though we have formulated this **H-CWAJ** correctly, and it fits the PRODAS simulated early flight data after the first 90 degrees of coning rotation, we do not see this horizontal aiming correction being of any practical use to riflemen, especially when shooting to normal rifle ranges where the **H-CWAJ** has been cancelled out.

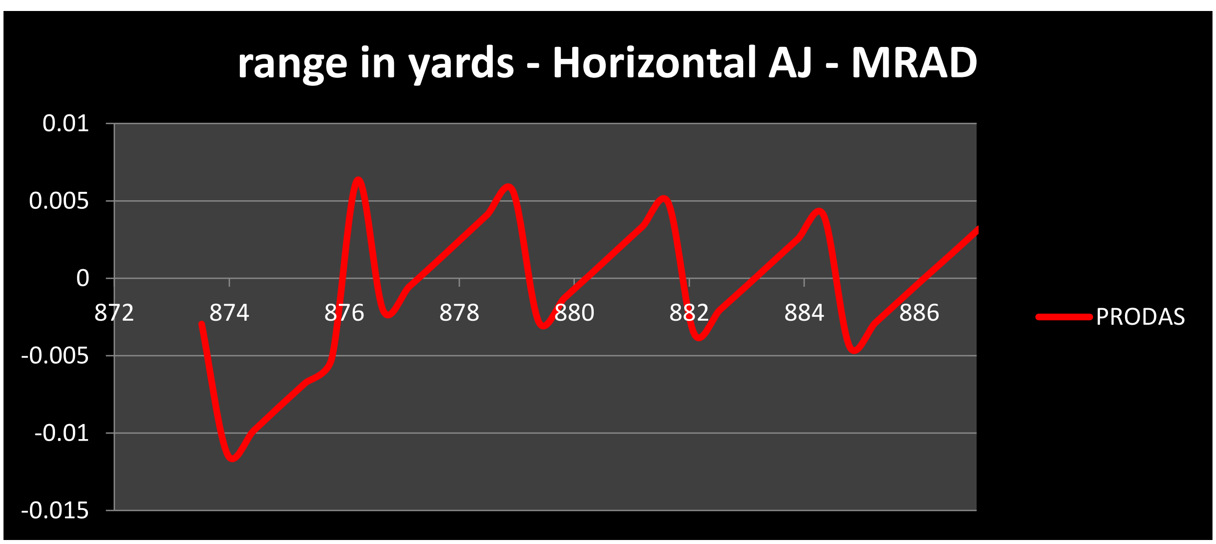
The following data from PRODAS 6-DoF runs confirm this in a definite manner, and as can be seen, as range increases, this horizontal perturbation tends to ***zero*** due to the angular cancellation effects of the rotating lift-force vector.



range in yards - Horizontal AJ - MRAD
0
-0.02
-0.04
-0.06
-0.08
-0.1
0
5
10
15
20
25
PRODAS



range in yards - Horizontal AJ - MRAD
0.005
0
-0.005
-0.01
-0.015
-0.02
197
202
207
212
217
222
PRODAS



range in yards - Horizontal AJ - MRAD
0.01
0.005
0
-0.005
-0.01
-0.015
872
874
876
878
880
882
884
886
PRODAS

Sample CWAJ Numerical Calculation

We shall calculate the Crosswind Aerodynamic Jump (**CWAJ**) aiming corrections for firing the well-known **308-caliber 168-grain Sierra MatchKing** bullet at **2800 feet per second** from a barrel having a right-hand twist-rate **Tw** of **+12 inches/turn** and in a **10 MPH (14.67 fps)** left-to-right horizontal wind from **9:00 o'clock** at the firing position.

The atmosphere at the firing point is an ICAO standard dry, sea-level atmosphere at 59 degrees Fahrenheit, or 15 degrees Celsius. The relative humidity is zero percent and the true barometric pressure at the firing point is 29.92 inches of mercury (inHg), or 760 millimeters Hg.

The air density **rho** for this dry ICAO standard atmosphere is **0.0764742 pounds per cubic foot.** Only a colder, higher barometric pressure, or below-sea-level atmosphere would be more dense than this standard.

We calculate the dynamic pressure **q** for this muzzle velocity V_0 as:

$$q = (rho/2)*V_0^2 = 64.704 \text{ lb/in}^2$$

The cross-sectional area **S** of this **0.3080-inch** diameter bullet (**d = 0.3080 inch = 1.00 "calibers"**) is:

$$S = (Pi/4)*d^2 = 0.074506 \text{ in}^2$$

The maximum potential aerodynamic drag force **q*S** on a 308-caliber bullet (having $CD_0 = 1.0$) fired at this muzzle velocity V_0 through this rather dense sea-level ICAO atmosphere is calculated as:

$$q*S = 4.821 \text{ pounds}$$

By convention, this subsonic potential drag force formulation is retained in supersonic aeroballistics to handle variations in air density **rho** correctly, even if the V^2 dependence does not quite hold true in that flight regime. The shape of the drag curve, CD_0 versus Mach-speed **M**, re-establishes the correct velocity dependence for supersonic flight.

The "speed of sound" **a** for this standard ICAO atmosphere is **1116.45 feet per second** (or **340.294 meters per second**). The correction of **a** for varying ambient temperature **T** (in Fahrenheit degrees) is always worthwhile:

$$a = 49.0223*SQRT(T + 459.67)$$

The Mach-speed **M** of the bullet at launch is then:

$$M = V_0/a = 2800 \text{ fps}/1116.45 \text{ fps} = 2.508$$

Referring to the data on this 30-caliber, 168-grain Sierra MatchKing bullet and dividing each linear dimension by **d = 0.3080 inch**:

L = 3.945 calibers

LN = 2.240 calibers

LBT = 0.455 calibers

DB = 0.786 calibers

DM = 0.211 calibers

RT/R = 0.900

BC(G7) = 0.223 (for 2500 fps < V_0 < 3000 fps).

Here we will use the gyroscopic stability **Sg** value for this bullet fired in "normal conditions" from a **12-inch** twist-rate, right-hand twist barrel:

Tw = +12.0 (inches per turn, from a right-twist barrel)

Sg = 1.74 (input data value).

Referring to the above table of zero-yaw coefficients of drag for the G7 Reference Projectile **CDref7** yields the interpolated result:

CDref7 = 0.270 (for Mach 2.508).

And, we can back figure our zero-yaw coefficient of drag value **CD_0** for this 168-grain bullet as:

CD_0 = (CDref7)*[Wt/ (7000*d^2)]/BC(G7) = 0.3063.

BRL measured the zero-yaw **CD_0** at Mach 2.5 for the somewhat similar 168-grain Sierra International bullet to be **0.32**, or 4.5 percent greater drag. That obsolete bullet had a larger meplat diameter **DM** which would account for its increased drag. That larger drag value corresponds to a **BC(G7)** value of just **0.214** for that earlier bullet.

To calculate the "interim estimate" of the coefficient of lift **CL** for this 168-grain Sierra MatchKing bullet, we refer to its nose dimensions in "caliber" units:

RT = [LN^2 + ((1 – DM)/2)^2]/(1 – DM) = 6.5567 calibers

LFT = SQRT(RT – 0.25) = 2.5113 calibers

LFC = LN/(1 – DM) = 2.8390 calibers

LFN = LFT*RTR + LFC*(1 – RTR) = 2.5441 calibers

B = SQRT (M^2 – 1) = 2.300

$CL = 1.974 + 0.921*(B/LFN) = 2.807$

BRL measured the coefficient of lift **CL** for the 168-grain Sierra International bullet to be **2.85** at Mach 2.5, or 1.5 percent larger. No boat-tail reduction of this **CL** estimate is needed for this bullet because it does not have an aerodynamically effective boat-tail design. Its boat-tail angle of 13.2 degrees is far too steep, so it flies as a 0.455-caliber shorter, flat-base bullet with extra mass hanging in its turbulent wake.

To calculate the yaw-drag coefficient **CDa** for this bullet, we first calculate its side-view aspect ratio **AR** in calibers:

$$AR = L - (2/3)*[LFN + LBT*(1 - DB)] = 2.1840$$

We estimate the yaw-drag adjusting coefficient **CDa** for this bullet as:

$$CDa = 1.33*(1.41 - 0.18*M)*[9.825 - 3.95*M + (0.1458*M - 0.1594)*CL^2*AR] = 4.4212$$

BRL measured this secondary yaw-drag adjusting coefficient at Mach 2.5 for the similar Sierra International bullet to be **4.4**, essentially the same as this estimate, and certainly well within the expected 25 percent (1 sigma) estimation accuracy.

We estimate the ratio of this bullet's second moments of inertia about its principal inertial axes **Iy/Ix** to be:

$$LL = L - LN + LFN = 3.945 - 2.240 + 2.5441 = 4.2491 \text{ calibers}$$

$$h = LFN/LL = 2.5441/4.2491 = 0.59874$$

$$\rho_p = 2750 \text{ gr/in}^3 \text{ (for this thin-jacketed, soft lead-cored match-type rifle bullet)}$$

$$Wtcalc = (\pi/4)*\rho_p*d^3 *LL*(1 - 2*h/3) = 161.113 \text{ grains}$$

$$f_1(LL,h) = 15 - 12*h + LL^2 *(60 - 160*h + 180*h^2 - 96*h^3 + 19*h^4)/(3 - 2*h) = 113.648$$

$$Iy/Ix = (Wt/Wtcalc)^{0.894} *f_1(LL,h)/[30*(1 - 4*h/5)] = 7.5482$$

BRL measured **Iy** and **Ix** for the similar International bullet, and their ratio **Iy/Ix = 7.44**, or 1.43 percent less than our estimation here for the 30-caliber 168-grain Sierra MatchKing. Bullets incorporating hollow cavities or significant low-density polymer components might require a slight downward adjusting of this **Iy/Ix** estimate.

We are now ready to calculate the tri-cyclic rates for our bullet's gyroscopic motions. First, we calculate the spin-rate **p** of the bullet as it exits the muzzle:

$$p = (12 \text{ inches/foot})*(V_0/ABS(Tw)) = 2800 \text{ rev/second} \qquad \text{(or 2800 hertz)}$$

Here, we are treating all frequencies in hertz as inherently non-negative values.

Next, we calculate the epicyclic ratio **R** of the gyroscopic nutation rate f_1 in hertz to its precession rate f_2 in hertz based on our expected value of the gyroscopic stability **Sg = 1.74**:

R = 2*{Sg + SQRT[Sg*(Sg – 1)]} – 1 = 4.75

We calculate the integer number **n** of fast-mode nutation cycles needed to establish steady coning motion as:

n = INT[(R – 1)/4] + 1 = INT[3.75/4] + 1 = 1

That is, only the *first* relative nutation cycle will be needed here.

Then, we find the sum of the two gyroscopic rates, f_1 and f_2 in hertz, as:

$f_1 + f_2 = f_2$*(R + 1) = p/(Iy/Ix) = 394 Hz

and

$f_2 = (f_1 + f_2)$/(R + 1) = 68.5 Hz

f_1 = 394 – 68.5 = 325.5 Hz.

We can now calculate the period T_N of the first relative nutation cycle of the spinning bullet and the period T_2 of the first slow-mode coning cycle:

T_N = 1/(325.5 – 68.5) = 3.891 milliseconds

$T_2 = 1/f_2$ = 1/68.5 Hz = 14.600 milliseconds

We calculate the initial crosswind angle **GAMMA** off the nose of the bullet as:

GAMMA = -14.67/2800 = -5.239 milliradians

From Coning Theory, the initial magnitude of the epicyclic swerve (precession and nutation) is:

GAMMA*R/(R – 1) = -5.239 mrad*(4.75/3.75) = -6.6364 milliradians

We can now calculate the yaw-drag adjusted total coefficient of drag **CD**:

CD = CD_0 + (-6.6364 milliradians)^2*CDa

CD = 0.3065

The “average” vertical **PITCH** attitude of the bullet during this first **n** relative nutation cycles is:

PITCH = GAMMA*[(R^2 – 1)/(n*2*Pi*R)]*[1 – Cos(n*2*Pi/(R – 1))] = -4.1799 mrad

The vertical impulse **Jv** is calculated as:

Jv = SIGN(Tw)*(n*T_N)*(q*S)*(CL + CD)*Sin(PITCH) = -0.00024413 pound-seconds

The initial “downwind” horizontal impulse **Jh** is calculated as:

Jh = - (T_2/4)*(2/Pi)^2 *GAMMA*SQRT [(R + 1)/(R – 1)]*(q*S)*(CL + CD)

= 0.000144020 pound-seconds

The magnitude **MOM** of the bullet's linear momentum vector is found from:

MOM = [Wt/(7000*g)]*V_0 = 2.0886 lb-sec

Finally, the needed elevation **EL** and windage **WI** aiming adjustments in milliradians and MOA, needed to compensate for this **CWAJ** effect are:

EL = -1000*Jv/MOM = -1000*(-0.00024413 lb-sec)/ 2.0886 lb-sec

EL = +0.1169 milliradians or +0.402 MOA (Upward)

WI = -1000*Jh/MOM = -1000*0.00021984 lb-sec/2.0886 lb-sec

WI = -0.0690 milliradians or -0.237 MOA (Leftward).

This **WI** aiming correction is calculated here only for academic purposes. Since this value rapidly decays exponentially with time-of-flight at an unspecified rate, it is ***not*** to be used as an aiming correction by riflemen in the field.

Comparison with Litz Vertical-AJ Estimator

Bryan Litz has published a simple regression-based linear estimator for calculating the angular size of the vertical Y-direction **CWAJ** using only the initial gyroscopic stability **Sg** and length **L** of the bullet in calibers:

Y (in MOA/MPH of crosswind) = 0.01*Sg – 0.0024*L + 0.032

It's worth noting that a linear fit of known data can produce good fits, which is typical of good linear regression data-fitting, whenever the case being examined comes from within the data set originally used for the regression analysis. However, extrapolation of the regression fit to other situations outside the range of the data used in the regression is rarely as successful.

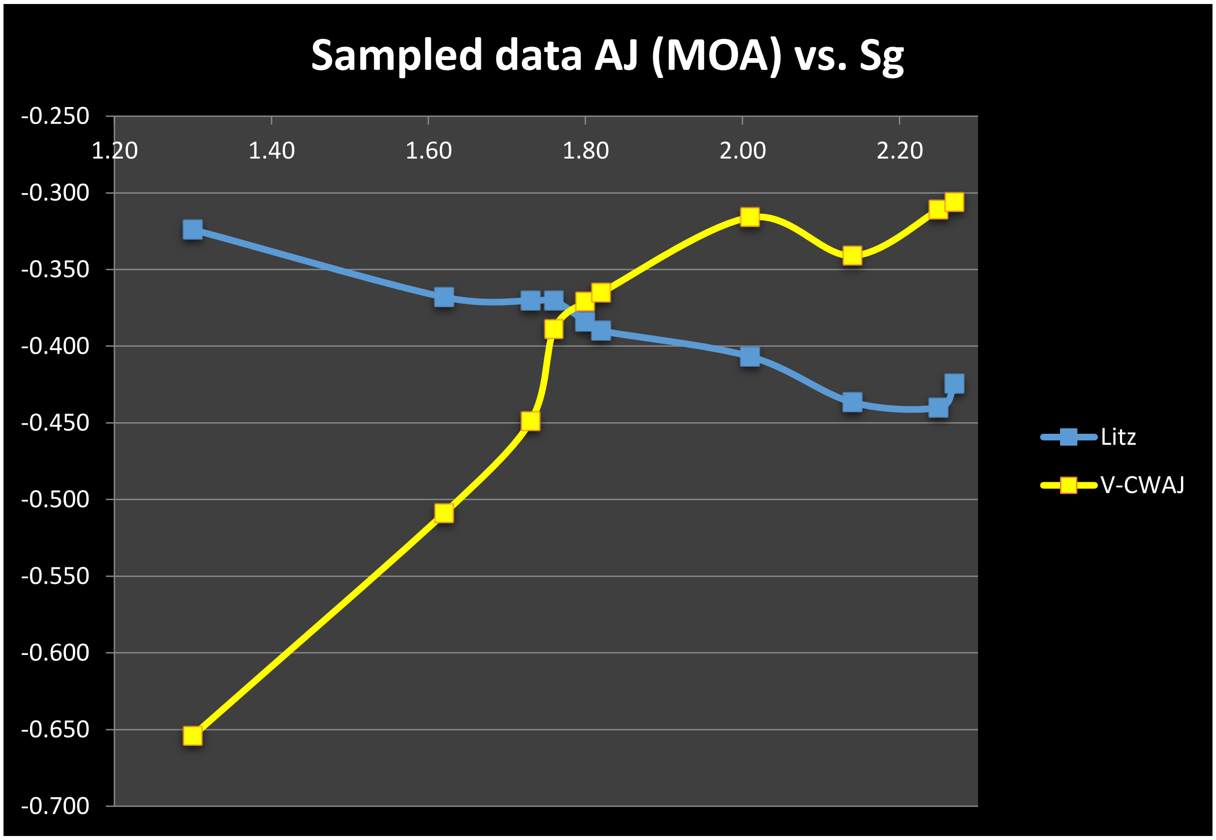
The following bullets were examined, and as expected, the divergence grows rapidly as the **Sg** (static stability) moves away from the (most likely) original bullet, the Sierra 168gr MK with a Sg of **1.75** under ICAO Std. Sea Level conditions. Check the graph below, with the bullets sorted by increasing **Sg** values, to see the high coincidence.

Bullet data: from the book "Applied Ballistics for Long-Range Shooting" by Bryan Litz.

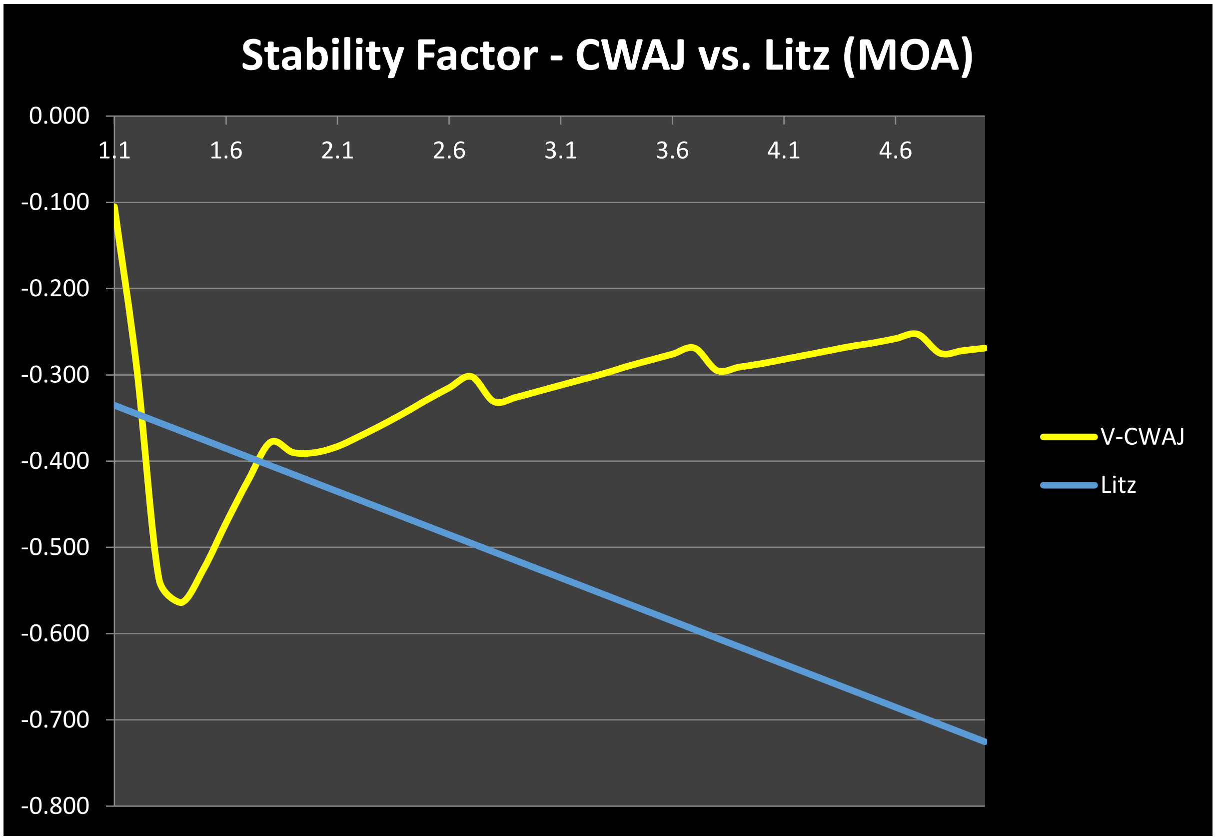
DATA - ICAO Sea Level Std. Conditions - Wind from 270° - Left to Right - Positive Twist											
Bullet	Bullet Diam	Bullet Length	Bullet Weight	Nose Length	BT Length	Base Diam	Meplat Diam	RTR	BC-G1	MV	Twist	Sg
Barnes .308 caliber 200 grain LRXBT	0.308	1.621	200	0.780	0.210	0.268	0.000	0.80	0.549	2900	10	1.30
Cutting Edge .308 caliber 180 grain HPBT	0.308	1.458	180	0.602	0.240	0.249	0.060	0.70	0.478	3000	10	1.62
Lehigh 408 caliber 400 grain Match Solid	0.408	2.085	400	1.155	0.320	0.326	0.000	0.78	0.759	2700	11	1.73
GS Custom 338 caliber 232 grain SP	0.338	1.771	232	1.036	0.346	0.238	0.020	0.60	0.604	3100	9	1.76
Sierra .308 caliber 220 grain MatchKing	0.308	1.489	220	0.672	0.230	0.234	0.070	0.95	0.607	2700	10	1.80
Nosler 270 caliber 140 grain Ballistic Tip	0.277	1.293	140	0.688	0.080	0.243	0.000	1.00	0.440	3100	9	1.82
Sierra .224 cal 80 grain MatchKing	0.224	1.066	80	0.629	0.135	0.183	0.060	0.98	0.425	3100	7	2.01
Berger 155.5 grain BT FULLBORE	0.308	1.250	155.5	0.825	0.160	0.264	0.062	0.96	0.464	2800	10	2.14
Berger .224 cal 70 grain VLD	0.224	0.976	70	0.471	0.150	0.177	0.052	0.53	0.371	3000	7	2.25
Hornady 338 caliber 285 grain BTHP Match	0.338	1.724	285	0.871	0.260	0.265	0.075	0.82	0.696	2800	9	2.27

This hypothesis is readily apparent when evaluating the following data and scatter diagram.

Litz	CWAJ-V	Difference %
-0.324	-0.654	-50
-0.368	-0.509	-28
-0.370	-0.449	-18
-0.370	-0.389	-5
-0.384	-0.371	3
-0.390	-0.365	7
-0.407	-0.316	29
-0.437	-0.341	28
-0.440	-0.311	41
-0.425	-0.306	39



The following chart shows how each method responds to a varying **Sg** (also called **stability factor**); it's clear that in the Litz method linear dependency on **Sg** is extremely high.



Therefore, if the bullet under study has a resulting **Sg** close to 1.75, the "sweet spot" resulting from the original fitted dataset, Litz's method can provide a fair estimate.

Otherwise the error grows very fast and this method becomes an unreliable AJ estimator, since it's a linear-bounded equation. This is likely to happen to any other oversimplified model.

Moreover, the Litz linear estimator does not exhibit, as it cannot, any response to the vertical and damped-out impulses (fast and slow arms alignment), thus not showing fidelity to the real phenomena.

The problem posed by the Litz estimator is that, neither deflection (horizontal or vertical) exhibits a linear correlation with respect to **Sg** in the aerodynamic behavior of spin-stabilized bullets under the influence of the aerodynamic jump.

This plot of **Sg**-sensitivity for our **V-CWAJ** formulation shows a small "ripple" at each step-change in the number **n** of fast-mode cycles being used. This simplification is made to allow more rapid firing solution calculations in field computers.

We could eliminate this artifact and improve the accuracy of our **CWAJ-V** algorithm by about 0.5 percent by using a full numerical integration for the vertical impulse **Jv** values over the first 180 degrees of the epicyclic motion of the spin-axis instead of using the closed-form solution available when the two arms of the epicyclic motion align at each cusp. We prefer the simpler approach for field computation. In some applications the numeric integration might be more suitable.

Testing case M118 Special Ball Mk 316 Mod (0)

Being a popular load, this ammunition displays high accuracy and is well known in the Long Range shooting community, so it appeared as a good candidate to test this **CWAJ** model against PRODAS 6-DoF and Litz' estimator.

All charts display downrange velocity (with its resultant **Sg** sway) vs. the resulting **Vertical Aerodynamic Jump** expressed in MOA.

The baseline comprises the bullet's flight over a range of 890 yards for a total ToF of 1.43 seconds, launched in windless conditions from an 11.5 inches/turn RH twist barrel at 2600 feet/second, under ICAO Sea-Level Standard Atmospheric Conditions.

PRODAS 6-DoF runs were set as the baseline for the different comparisons, since it's a proved de-facto standard for complex aeroballistics modeling. Also worth noting is that the aeroballistic coefficients are regularly derived from experimental data, where the instrumented data collection equipment (and procedures) is of high precision.

Caveat: neither **CWAJ** nor Litz were meant to calculate a continuous trajectory (both are static methods, intended to work only at the muzzle, whereinafter the aeroballistic coefficients are constantly changing).

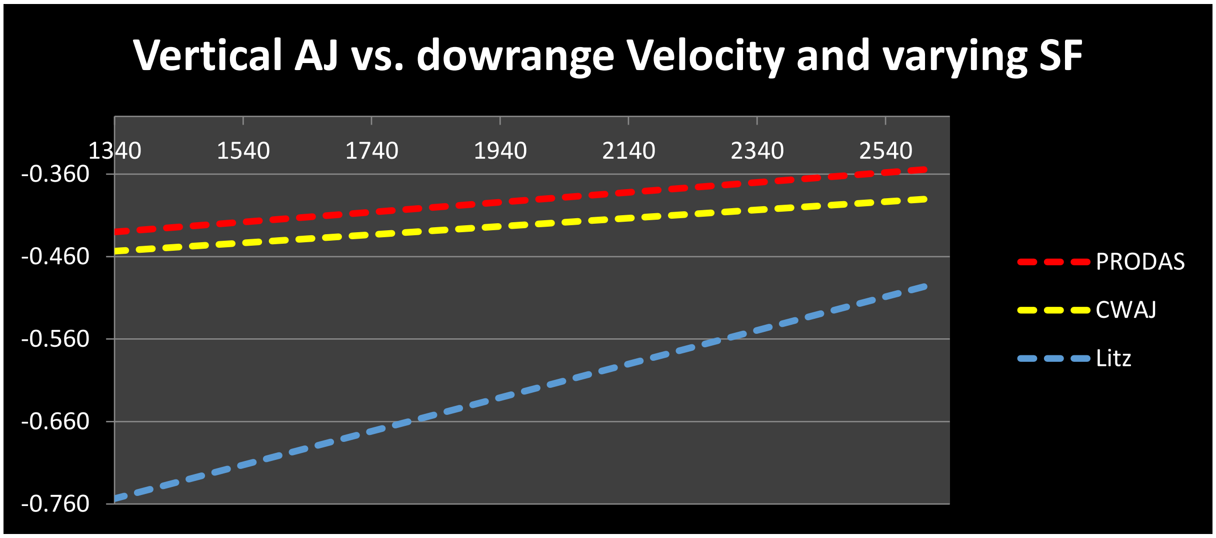
Nonetheless, the rationale for this extension was the necessity to understand how stable, how resilient, and how robust they can be when subjected to a real trajectory (the baseline is indeed a dynamic breakdown), where change is the constant.

In other words it's a sort of stress-testing necessary to check under the extremes, usually at the boundaries where most conceptual formulation flaws are clearly exposed.

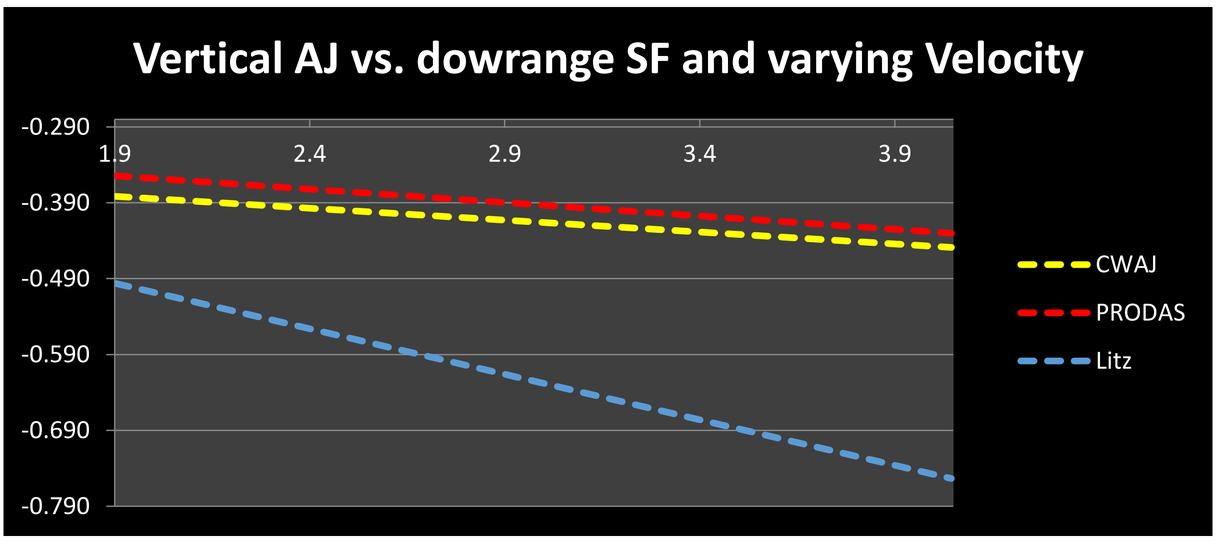
So, while the following charts do not represent the expected outcome of either **CWAJ** or Litz, how they behave under the influence of critical inputs such as velocity and **Sg,** is interesting to realize that even when using practical methods, if done right with proper physics, the resulting fidelity can be there or not, which is visualized how close the curves are to the baseline (vertical gap) as well as their respective slopes (horizontal gap) between the initial and final conditions.

Note: In order to eliminate rounding off (truncating) errors that will only cause an unnecessary misreading of the resulting plots, the results were linearized (dashed lines)

The following linearizations created some isotropic axial skewness, which is an unavoidable side-effect of the technique used given the discontinuous datasets under study, but the implication here, is to illustrate their development as range increases along with a decaying velocity and the resulting increase in **Sg**.



The Standard Deviation is 2.5% for the **CWAJ** model and 12.9% for the Litz's estimator, percentages that given the linearization remains the same for the next dataset.



Even when the recommended condition of **Sg ≈ 1.5** is far from being kept close, along the entire flight; the **CWAJ** model demonstrates its robustness through a wide **Sg** shift, spanning from 1.9 to 4.1

Sensitivity analysis

Sensitivity analysis was significant to study and assess the behavior of uncertainty in the output of our model, which can be attributed to different sources of error of the input parameters.

Sensitivity analysis is an integral part of model development and involves analytical examination of input parameters to aid in model validation and provide guidance for future research.

We needed it to determine how different values of one or more independent variables, impact a particular dependent variable under a given set of conditions.

In other words, it helped us to investigate the robustness of the model predictions and to explore the impact of varying input assumptions.

We chose to set on what is known as Local (sampled) sensitivity analysis, which is derivative based (numerical or analytical).

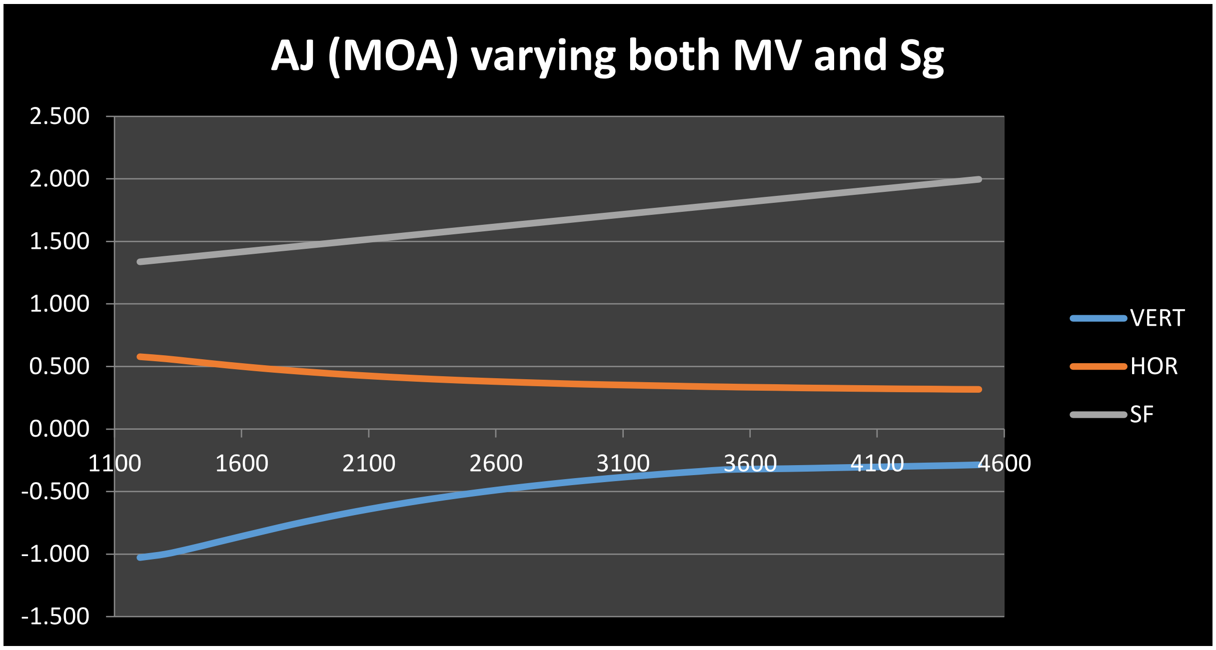
The use of this technique is the assessment of the local impact of input factors' variation on model response by concentrating on the sensitivity in the vicinity of a likely set of factor values.

Such sensitivity is often evaluated through gradients or partial derivatives of the output functions at these factor values, (the values of other input factors are kept constant) when studying the local sensitivity of a given input factor.

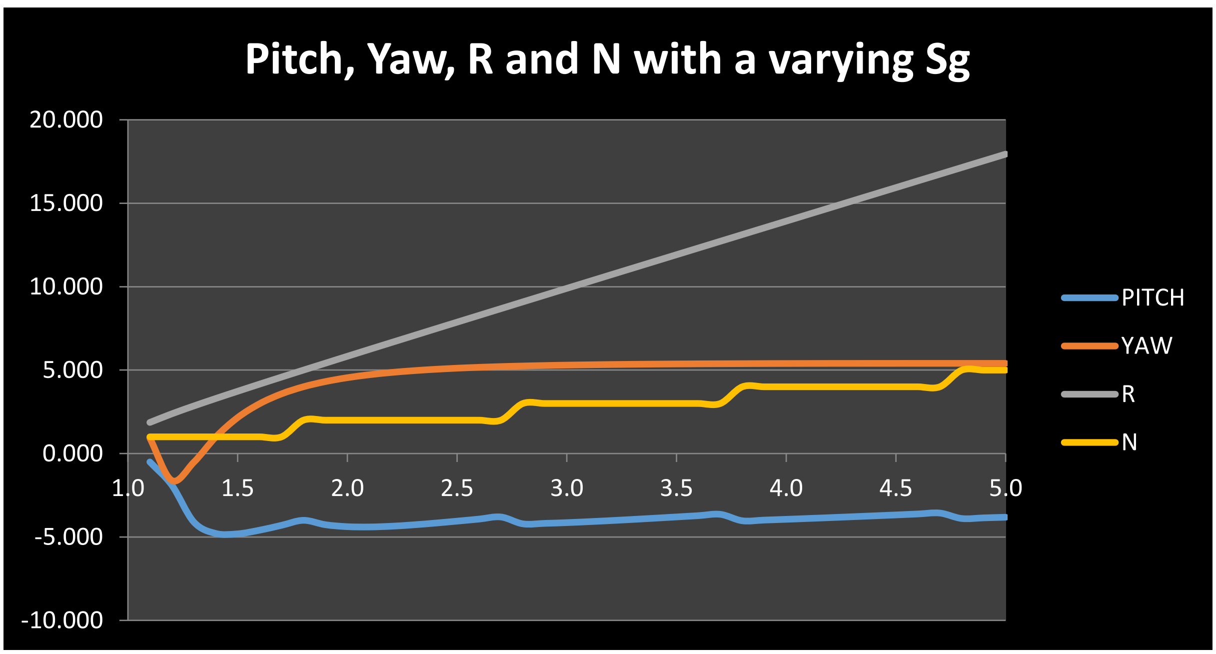
One of the critical objectives was to stress-test the model as well as to study its fidelity to known experimental and Equations of Motion-based 6-DoF runs.

Unfortunately there are many results and accompanying charts to add, but in order to make the document more manageable we’ve chosen to include only a pair of them, which are significant in terms of reliability of the underlying mathematical numerical algorithm.

The first chart shows how the Vertical and Horizontal deflections, due to the presence of a crosswind, behave under a scenario where Muzzle Velocity and Stability Factor (**Sg**) are varying and influencing each other as doing so.



The second one, where only **Sg** is varied, shows the effect it has on the Pitch, Yaw, R (ratio of the inertial gyroscopic nutation and precession rates) and **n** (initial fast-mode cycles).



Closing summary

Our sole goal and direction for this team effort, was to develop a practical yet highly accurate method to compute this complex phenomenon, since the problem itself is not prone to simple workarounds, making it very difficult to quantify by shooters given the required aeroballistics coefficients.

All sensitivity and baseline-comparisons have shown so far, that this novel approach can be relied upon as being a trustworthy tool for general use, not limited to be used only by expert ballisticians.

The **CWAJ** model had demonstrated, against experimental and 6-DoF runs, fidelity near or better than $\pm$ 2.5 percent, 1-Sigma, which highly satisfies our initial goals.

On the other hand, considering our purpose, we felt it important to compare this model against a known practical estimator, such as the one published by Bryan Litz, because as far as we are aware, no one else has presented a straightforward solution so far, besides his, hence the need arose as significant for the intended audience and the shooting community in general.

As an interesting historical footnote, it's to be noted that McCoy did not handle crosswinds correctly in his early 6-DoF program. He was working primarily with indoor spark-range firing data at BRL in those earlier days. He calculated an initial yaw vector due to the horizontal crosswind (W/V) but did not add the 3-dimensional vector difference (W – V) for the airstream effects at each time-step. This error is easily seen in "wind axes" plots of the bullet's spin-axis orientation whenever the epicyclic pattern is centered about the +V direction of bullet motion instead of about the approaching wind direction.

We want to thank all who have sent us their feedback and encouragement to complete this work.