Effect of Rifle Cant on Point of Impact at Low and High Inclinations

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Abstract

The relationship between the position of a gun's aiming device, such as a telescope, and the point of impact as the gun is tilted ("canted") about the line of sight is a simple yet intriguing problem. I discuss rifle cant in simple mathematical terms, with particular attention to very steep uphill and downhill firing, when straightforward application of the Rifleman's Rule breaks down. I give a rationale as to why the point of impact is almost independent of the vertical position of the aiming device as well as the reason why this independence breaks down when the gun is improperly zeroed at the target distance.

1.0 Background

The problem I will discuss is how the vertical distance of the aiming device above the rifle affects the point of impact (POI) as you tilt the gun by an angle called the angle of cant. I will refer to the distance between the aiming device above the gun and the axis of the barrel as the "sight height", regardless of whether the aiming device is a telescope, irons sights, or something else. I will assume that this tilting is a rotation about the line of sight (LOS), which you can visualize as the line that joins your aiming eye with your target. This problem belongs in a category I call "embarrassingly trivial problems". It is trivial because you can solve it with basic reasoning, and is embarrassing because even smart people often get it wrong. Problems of this kind tend to contradict one's intuition, which makes them entertaining to solve[1].

It is clear that canting your gun about the line of sight will cause the POI to migrate - this migration is referred to as "canting error". It is less clear how the *vertical position* of the aiming device above the gun affects the magnitude of this migration.

It will turn out that, assuming you gun is properly zeroed at your target distance, the influence of the sight height is irrelevant for practical purposes. This doesn't mean the sight height has no influence on the POI migration due to cant on a properly zeroed gun, what it means is that this influence is extremely small and you can neglect it.

1. Example: Assume a perfectly smooth sphere the size of Earth with an inextensible ribbon laid tightly around the equator. Now imagine you add one meter to the length of the ribbon, causing it to slack some. Assuming there is no gravity, by how much would the ribbon float off the surface of the sphere? Hint: Earth's circumference is 40 million meters.

The problem at hand is too simple to merit publishable research, and this is the reason I don't list any scientific references. There is, however, some internet "chatter" about this topic. One internet source worth noticing, which correctly describes much of the physics involved, is *High Scope and Canting, the End of an Ancient Myth,* by Andras Fekete-Moro. This source, however, doesn't identify the aspects that cause cant sensitivity to be influenced, albeit very slightly, by sight height even in the case of a correctly sighted gun, nor does it develop the math to correctly tackle the case of firing under extreme uphill or downhill slant.

Firing at very steep slant angles (especially uphill) is of interest in law enforcement situations (such as firing at a target on a tall building), and in specialized sports applications, such as Field Target, where a small target might be placed high above the ground. It will turn out that the crucial element in the mechanism of canting error is the elevation angle, the angle the line of sight makes with the exit velocity vector.

A challenge arises in getting a simple estimation of the elevation angle under slant firing that doesn't involve numerically solving the equations of motion. The standard way to accomplish this is by invoking the so-called Rifleman's Rule. The Rifleman's Rule says the elevation that zeroes a gun on a slanted range is the same as the elevation that zeroes the gun on the horizontal projection of the range. The Rifleman's Rule breaks down for a sufficiently steep slant in such a way that the inclination for which it breaks down depends on the sight height - the higher the sights, the sooner it breaks down.

2.0 Preliminaries

As mentioned above, cant consists in rotating the gun about the line that connects your aiming eye with your point of aim, or line of sight. In principle, you could rotate your gun about some other line, but such a rotation wouldn't be called *cant*. Your point of aim usually coincides with your target (intended point of impact), but it doesn't have to[1]. If your POA coincides with your intended POI, you achieve your goal of hitting your target by adjusting the elevation angle, which is the angle subtended by the LOS and the axis of the bore. In this case, if your gun is fitted with a telescope, you have two modalities to adjust your elevation. In one modality you adjust your elevation by "clicking" up or down, namely, by displacing the scope reticle. In another modality, you adjust your elevation by leaving the reticle fixed relative to the scope and moving the gun-plus-scope to get the desired sight picture (referred to as holdover or holdunder, depending on the adjustment direction). In either case, your gun may or may not be correctly zeroed at the target distance.

If your gun is properly zeroed at the target distance, the analysis will show that vertical and horizontal displacements of your point of impact due to cant are almost, but no quite, insensitive to sight height.

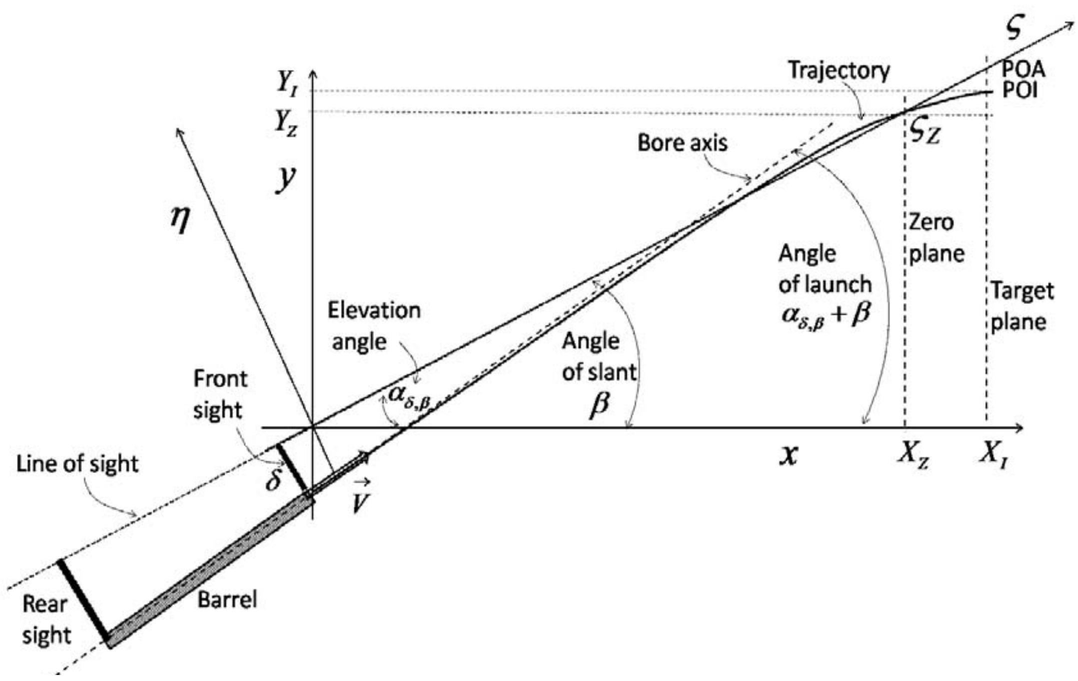
1. This situation occurs when your aiming device runs out of elevation adjustment, as with older military rifles whose minimum elevation, set at 300 meters, is too high for modern circumstances, compelling the shooter to aim lower than his intended point of impact

2.1 Parameters and layout

Figure 1 shows the parameters and their definitions in the reference configuration (the gun position in the absence of cant.) The gun is represented by its barrel and its front and rear iron sights. Doing the analysis with iron sights - as opposed to a scope or other aiming device - is convenient for two reasons. One is that there is no ambiguity about the position of the sights, the other is that the sight height is clearly defined. These two quantities, position and height, are also properties of scopes, but their precise values are harder to determine. In the case of a telescope, it is best to regard the sight height as a quantity to be determined indirectly. In the Appendix I discuss the details of how you can use the properties of a parabola to accomplish this task.

The basic assumption about cant is that the reference configuration has a vertical plane of symmetry. It is convenient to use two orthogonal systems of coordinates to deal with uphill or downhill shooting. In the x, y system, the x-coordinate, coincident with the LOF, is horizontal and the origin is such that the vertical axis intercepts the bore axis in the muzzle plane. The η, ζ results from rotating the x, y system counterclockwise by the angle of slant. This means the ζ coordinate, also coincident with the LOS, is parallel to the slant. In addition the x, y pair of coordinates there is a third coordinate, z, perpendicular to x, y, pointing out of the page. We won't need the z coordinate in our analysis.

FIGURE 1. Layout and definition of parameters - the *x*-coordinate is parallel to the horizontal and the y-coordinate lines up with the direction of gravity, the ζ coordinate line up with the LOS. The zero plane and the target plane are orthogonal to the x,y plane and perpendicular to the horizontal.



If the target is not at the same level with respect to the horizontal as the top of the front sight, the firing occurs on a slant (uphill or downhill). The angle of slant, β, is the angle the line of sight makes with the horizontal. The elevation angle, denoted by $\alpha_{\delta,\,\beta}$, where the subscripts refer to the values of the sight height and slant angle, is the angle the LOS makes with the axis of the bore. This angle is positive as shown in Figure 1. The angle the bore makes with the horizontal is called the angle of launch. This angle equals the sum of the elevation angle and the angle of slant.

For each elevation angle and each slant angle there is a point defined by X_Z, Y_Z for which the gun is zeroed. This point is contained in the *zero plane*. The POI, defined by X_I, Y_I is in the *target plane*, which may lie forward or aft of the zero plane. ζ_Z is the zero distance along the slant. Both the zero and the target plane are vertical to the horizontal. If the target plane and the zero plane coincide, the gun is zeroed at the target distance. Notice that the assumption of symmetry in the reference configuration implies there is no windage aiming error other than canting error. In other words, while the shooter may misjudge the target distance, she doesn't misjudge the lateral position of the target. The reason for this assumption is that I want to isolate the impact of canting error from other sources of windage error. For the same reason, I assume there is no wind.

3.0 Trajectories and cant

The angle of cant, denoted by φ, is positive for a clockwise rotation when looking down the line of sight. Capturing the effect of cant occurs automatically when you compute the projectile trajectory, since the angle of cant enters in the initial conditions. However, the fact you can *calculate* the effect of canting error very accurately does not mean you can gain an *understanding* of canting error through such calculations. As with any numerical computation, the effect you are interested in will lie buried in the numerical output and will be hard to extricate.

What you want is to formulate the problem with enough simplicity that allows you to get an analytical representation of the issue, but with not too much simplicity that what you are trying to understand is washed away in the simplification process.

The vacuum trajectory, where the projectile is not subjected to drag, is a good proxy to capture the essential elements of cant with minimal mathematical complexity. The parameter that captures cant sensitivity is the elevation angle *in the absence of cant*. The reason why you need the elevation angle in the absence of cant is rooted in the way cant occurs in practice - the marksman aims his gun first, and then he cants his gun, either inadvertently or intentionally. Notice that knowing the elevation angle is equivalent[1] to knowing the total drop the bullet would undergo during its flight to the target, assuming the gun were fired parallel to the ground.

1. This equivalence is actually an approximation that holds to the extent the rigid trajectory assumption holds, but it is extremely good in practice.

3.1 Elevation angle

Given the target distance and the exit velocity, the elevation angle is influenced by the sight height and the slant angle. In order to capture the effect of cant in a simple and intuitive way, it is convenient to separate the sight height from the elevation angle at any slant angle, and the latter from the elevation in horizontal firing[1]. Since the sight height is a length scale much smaller than the target distance, simple geometry says you can get the elevation angle for non-zero values of the sight height and slant angle by adding a correction to the elevation for zero sight height, $\alpha_{\delta,\beta} = \frac{\delta}{\zeta_Z} + \alpha_{0,\beta}$, where ζ_Z is the target distance along the slant. Since $\zeta_Z = \frac{X_Z}{\cos\beta}$,

$$\alpha_{\delta,\beta} = \frac{\delta}{X_Z}\cos\beta + \alpha_{0,\beta}, \qquad \text{(EQ 1)}$$

where the elevation for arbitrary slant is a function of the elevation in horizontal firing,

$$\alpha_{0,\beta} = f(\alpha_{0,0}) \qquad \text{(EQ 2)}$$

If you invoke the rigid trajectory assumption, which consists in assuming the trajectory responds to the elevation angle like a slender, rigid rod, the elevation angle is simply $\alpha_{0,0} = (\text{Bullet drop at } X_Z)/X_z$, where the drop is with respect to the horizontal. Using the vacuum trajectory, if you assume the projectile exits horizontally, it will travel with a constant horizontal velocity component V and its drop at X_Z will be $\frac{1}{2}g\left(\frac{X_Z}{V}\right)^2$, where g is the acceleration of gravity. Invoking the rigid trajectory assumption, the elevation angle to bring the POI to the horizontal is[2],

$$\alpha_{0,0} = \frac{1}{2}g\frac{X_Z}{V^2} \qquad \text{(EQ 3)}$$

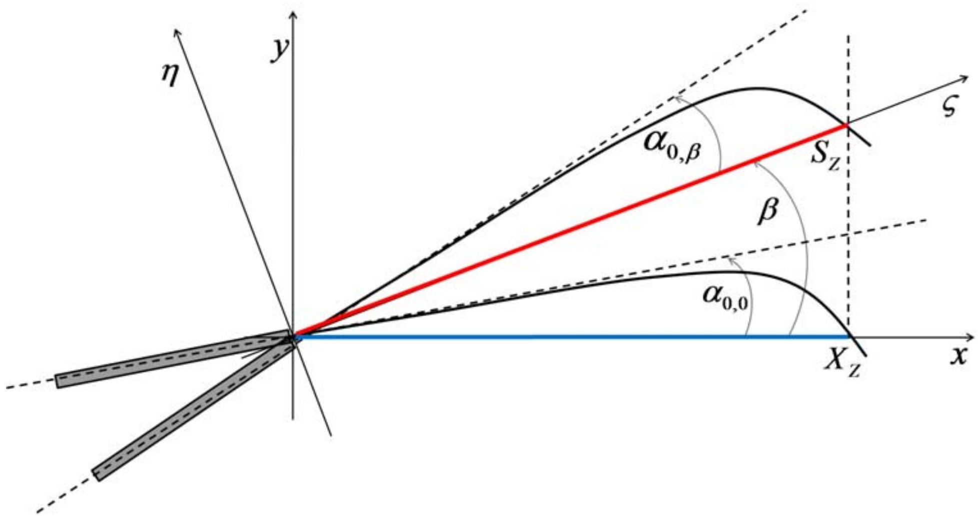
Getting a formula for the elevation angle for an arbitrary slant angle isn't a straightforward extension of the derivation behind EQ 3, however. The reason is that some assumptions behind the formula for $\alpha_{0,0}$ break down when β is large. The rigid trajectory assumption,

1. Working with the equations of motion you can get an exact formula for the elevation angle consistent with Figure 1, namely, $\alpha_{\delta,\beta} = \operatorname{atan}\left\{\frac{V^2}{gX_Z}\left[1 \pm \sqrt{1 - \frac{2g}{V^2}\left(X_Z\tan\beta + \delta\frac{\cos\alpha_{\delta,\beta}}{\cos\beta} + \frac{1}{2}g\frac{X_Z^2}{V^2}\right)}\right]\right\} - \beta$. However, this expression is algebraically less friendly than the approach I will follow here.
2. This result is not exactly the same as what you would obtain by enforcing the POI position in the drag-free solution of the equations of motion, but it is extremely close.

for example, breaks down when the trajectory is extremely "loopy", which is the case when the angle of slant is large[1].

A simple and intuitive strategy to get $\alpha_{0,\beta} = f(\alpha_{0,0})$ is to solve the equations of motion in two coordinate systems, x, y and ζ, η, illustrated in Figure 2.

FIGURE 2. Elevation angle at varying angle of sight for zero sight height.



Solving the vertical equation of motion in the ζ, η coordinates and the horizontal equation of motion in the x, y coordinates gives us the system,

$$\left.\begin{aligned} \frac{V^2}{g}[\sin(2\alpha_{0,\beta}) - 2\tan\beta\sin^2\alpha_{0,\beta}] = \zeta_Z\cos\beta \\ \frac{V^2}{g}\sin 2\alpha_{0,0} = X_Z \\ \zeta_Z\cos\beta = X_Z \end{aligned}\right\} \tag{EQ 4}$$

from which,

$$\alpha_{0,\beta} = \operatorname{asin}\left\{\frac{1}{2}\frac{\cos\alpha_{0,0}}{\tan\beta}[1 - \sqrt{1 - 4\tan\beta\tan\alpha_{0,0}}]\right\}. \tag{EQ 5}$$

Since in all practical cases of interest in small arms the elevation angle is much smaller than one radian,

1. Large in this context means significantly close to 90 deg, depending on the sight height.

$$\alpha_{0,\,\beta} \;=\; \frac{1}{2}\cot\beta\left[1-\sqrt{1-4\tan\beta\,\alpha_{0,\,0}}\right] \qquad \text{(EQ 6)}$$

If, in addition, $4\tan\beta\,\alpha_{0,\,0} \ll 1$, a first-order expansion gives us the first order relationship,

$$\alpha_{0,\,\beta} \cong \alpha_{0,\,0} \qquad \text{(EQ 7)}$$

This approximate equality is remarkably good even for large angles of slant. To see why this is the case, imagine your target is at a fixed height, h. As you approach the target plane along the horizontal, $\alpha_{0,\,0}$ will approach the value $\frac{1}{2}g\frac{h\cos\beta}{V^2}$, which means $4\tan\beta\,\alpha_{0,\,0}$ is bounded from above by $2g\frac{h}{V^2}$. In metric units, for a high-power rifle and a target on a building 100 meters high, $2g\frac{h}{V^2} \;=\; O\left(20\times\frac{100}{1000^2}\right)$, which is of order 2×10^{-3}. It is important to realize that this very tight approximation, $\alpha_{0,\,\beta} \cong \alpha_{0,\,0}$, strictly applies when the sight height is zero. This means the condition $4\tan\beta\,\alpha_{0,\,0} \ll 1$ is necessary, but not sufficient, for the Rifleman's Rule to apply. As mentioned earlier, the Rifleman's Rule says that if the angle of slant isn't too large, the same elevation that zeroes your gun on a horizontal range will also zero your gun on an uphill or downhill range. However, the elevation angle when the sight height is not zero contains a correction that doesn't remain invariant with respect to the slant angle under the same conditions where $\alpha_{0,\,\beta} \cong \alpha_{0,\,0}$ holds.

FIGURE 3. Elevations versus angle of slant for two sight heights, firing from a horizontal distance of 5 m, for an Air Rifle (AR) and a High Power Rifle (HPR) with exit velocities 250 m/s and 850 m/s, respectively.

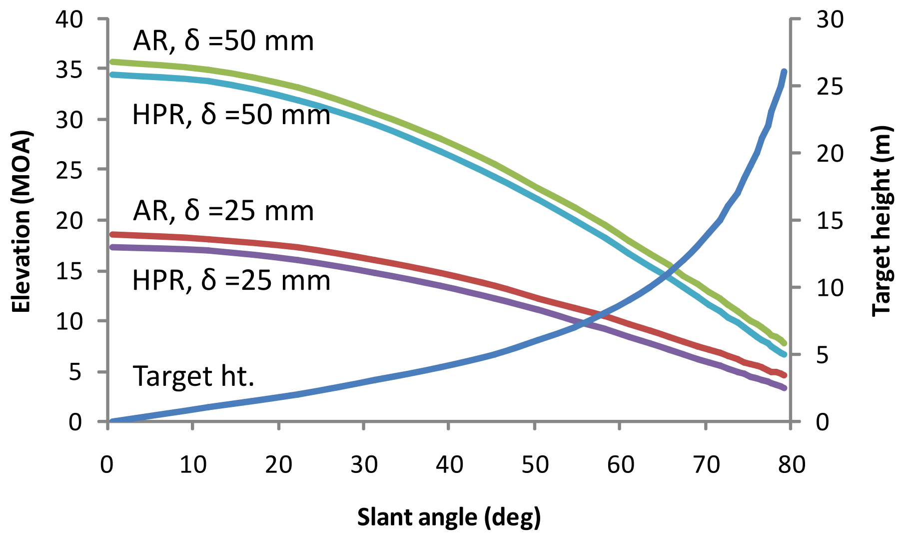


Figure 3 shows the elevations you need to zero on a target at variable height, plotted as a function of the angle of slant, for an air gun and a high-power rifle, each with sight heights of 25 mm and 50 mm. The figure shows values for uphill firing. For downhill firing the elevation curves are almost symmetrical, and the height curve is perfectly anti-symmetrical. The interval of slant angle where these curves remain approximately horizontal are regions where the Rifleman's Rule applies. Both in the airgun and in the high-power rifle, this region is up to about 35 degrees for the 25 mm sight and up to about 20 degrees for the 50 mm sight - the higher the sights, the narrower the range of validity of the Rifleman's Rule.

4.0 Cant sensitivity

When the gun is canted, the POI undergoes a displacement with a horizontal and a vertical component in the target plane. The horizontal and vertical cant sensitivities are the horizontal and vertical POI displacements per unit of cant rotation. Of the two, the horizontal displacement is far more significant. Therefore, unless it is explicitly indicated otherwise, cant sensitivity refers to the horizontal component of the POI displacement.

4.1 Horizontal firing

You can use a simple geometrical argument to illustrate how cant causes the POI to migrate. To construct this argument, assume you can turn off the effect of gravity and observe the POI loci as you rotate the rifle about the line of sight. Then assume you turn gravity on, and observe what happens to the POI loci as a result of the bullet drop.

For clarity, consider first the case of zero sight height and assume the target distance and the zero distance coincide. As discussed in the previous section, $\alpha_{0,\,0}$ denotes the elevation angle to zero your gun at distance X_Z when there is no front sight (equivalently, when the sight height is zero) and when the slant angle is zero. Figure 4 shows what would happen to the POI if you turned off the effect of gravity and rotated the gun about the line of sight (which in this case makes an angle $\alpha_{0,\,0}$ with the bore axis). The bore axis intercept with the target plane would describe a circle of radius R, and the effect of cant would be to move the POI horizontally by $R\sin\varphi$ and in the vertical direction by $R\cos\varphi$.

Now assume gravity is turned on. The POI trace is displaced downwards by the bullet drop. Gravity will displace the circle in Figure 4 downward, a*lmost* without distortion, such that the uppermost point of the circle coincides with the point of aim. At this time I am assuming the target distance coincides with the zero distance, I will relax this requirement soon. This is illustrated in Figure 5. Assuming the circle doesn't get distorted in its displacement, the vertical and horizontal components of POI position due to cant are,

$$\begin{bmatrix} Y \\ Z \end{bmatrix} = R\begin{bmatrix} \cos\varphi - 1 \\ \sin\varphi \end{bmatrix} \qquad \text{(EQ 8)}$$

which is the parametric representation of a circle. The vertical and the horizontal changes in POI position for a small value of the cant angle are,

$$\begin{bmatrix}\Delta Y\\ \Delta Z\end{bmatrix} = R\begin{bmatrix}-\frac{1}{2}\varphi^2\\ \varphi\end{bmatrix} + \begin{bmatrix}O(\varphi^3)\\ O(\varphi^2)\end{bmatrix} \qquad \text{(EQ 9)}$$

Notice that the vertical sensitivity for small angles of cant is $O(\varphi^2)$, which implies that the ratio of lateral to vertical deviations due to cant is of order $O\left(\frac{\varphi}{2}\right)$. *This means that for a five-degree cant, the vertical drop of the POI is over 20 times smaller than the lateral deviation. For a one-degree cant, the vertical drop is over 100 times smaller.* This justifies the common assumption that cant only affects the lateral position of the POI, and has a negligible effect on the vertical position.

You can compute R by identifying it with the bullet drop at the target location assuming the gun is fired horizontally (with zero elevation), or by computing $\alpha_{0,0}$ and using the relationship,

$$R = X_Z \tan\alpha_{0,0} \cong X_Z\alpha_{0,0} \qquad \text{(EQ 10)}$$

These two views are equivalent to first order in $\alpha_{0,0}$. You can also get R from a drop table by scaling the drop values (this is sometimes provided by the ammo manufacturer). Using the vacuum trajectory as a proxy, $\alpha_{0,0} = \frac{1}{2}g\frac{X_Z}{V^2}$. Therefore, the cant sensitivity under the vacuum trajectory for a gun zeroed at X_Z, assuming the $X_I = X_Z$, is

$$R = \frac{1}{2}g\left(\frac{X_Z}{V}\right)^2 \qquad \text{(EQ 11)}$$

FIGURE 4. Trace of barrel axis intercept with target plane as barrel rotates about line of sight axis, assuming there is no front sight and the target plane agrees with the zero plane.

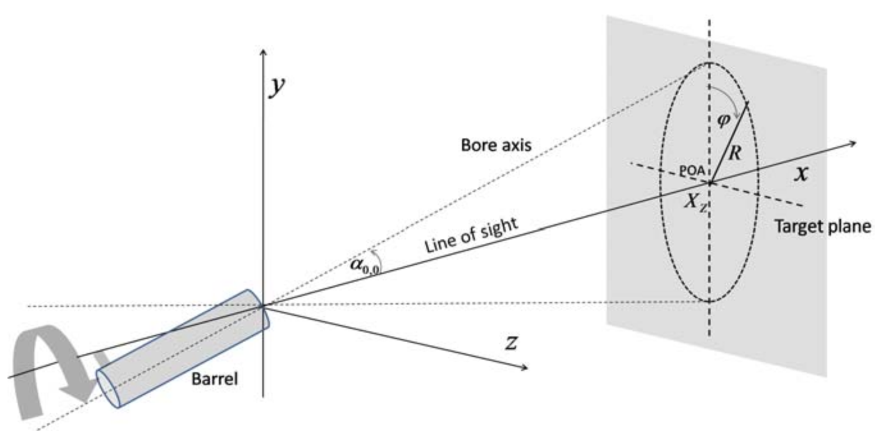
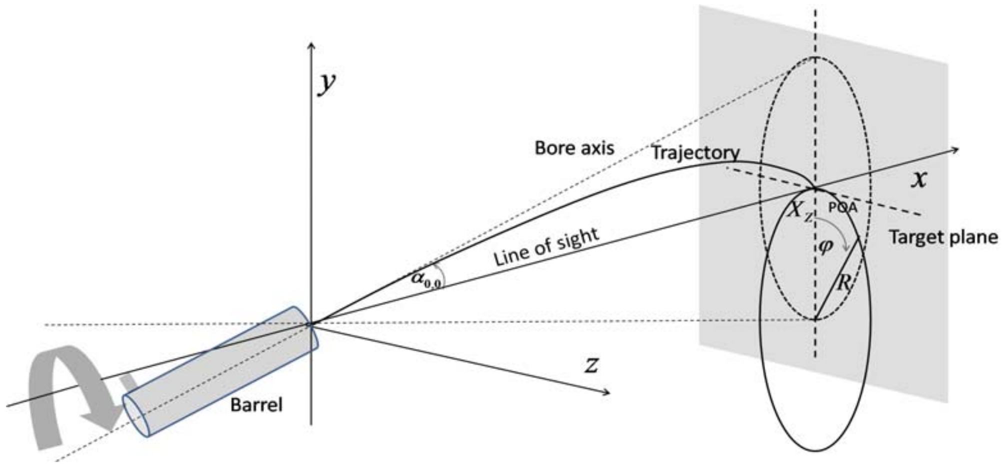
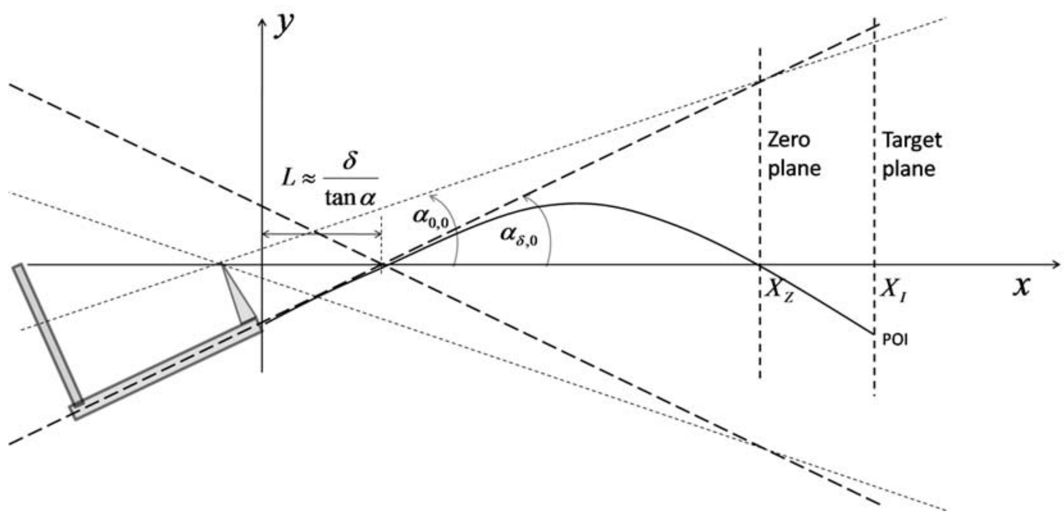


FIGURE 5. Effect of gravity on circle shown in Figure 4.



Let's now add a front sight. Figure 6 shows a lateral view in the x, y plane of the line of sight and bore axis intercepts with the zero plane and the target plane. In the case of a properly zeroed gun the zero plane and the target plane coincide, but in general these two planes are not the same. In the presence of a front sight, the bore axis describes a double cone as the gun rotates about the line of sight. The approximate position of the vertex of this double cone from the muzzle is $L \cong \dfrac{\delta}{\tan\alpha_{0,\,0}}$.

FIGURE 6. Double-cone geometry when the gun is fitted with a front sight (or a scope) and is rotated about the line of sight.



The cant sensitivity in the case of sight height δ is,

$$R = -\delta + \alpha_{\delta,\,0} X_I \qquad \text{(EQ 12)}$$

Replacing $\alpha_{\delta,0} = \frac{\delta}{X_Z} + \alpha_{0,0}$ from the previous section[1],

$$R = \delta\left(\frac{X_I}{X_Z} - 1\right) + \alpha_{0,0}X_I \qquad \text{(EQ 13)}$$

In the case of the vacuum trajectory,

$$R = \delta\left(\frac{X_I}{X_Z} - 1\right) + \frac{1}{2}g\frac{X_Z X_I}{V^2} \qquad \text{(EQ 14)}$$

Notice the cant sensitivity depends explicitly on the front sight height. *This dependence disappears when the impact distance is the same as the zero distance*. Notice the dependency on sight height component switches sign depending on whether the impact plane is behind or in front of the zero plane.

4.2 Shooting on a slant

When shooting uphill or downhill, the line of sight is at an angle β, referred to as the slant angle, with respect to the horizontal. By convention the slant angle is positive when shooting uphill and negative when shooting downhill. Figure 7 shows the extension of the geometric argument to this case. Notice that the target plane is assumed to be perpendicular to the horizontal.

Just as in horizontal shooting, the analysis of cant sensitivity hinges on getting a proper assessment of the elevation angle, which is always measured with respect to the line of sight. If the angle of slant is moderately small, which is usually the case in most situations, you can calculate the elevation angle with satisfactory accuracy resorting to the so-called Rifleman's Rule. This, plus a little trigonometry, is all you need to compute the cant sensitivity in uphill or downhill firing when the slant angle is moderately small. However, in cases where the angle of slant is very large the standard Rifleman's Rule is not sufficient for an accurate assessment. This would be the situation when firing at a target on top of a tall building a short distance away from the base of the building or in Airgun Field Target when the target is perched high in a tree.

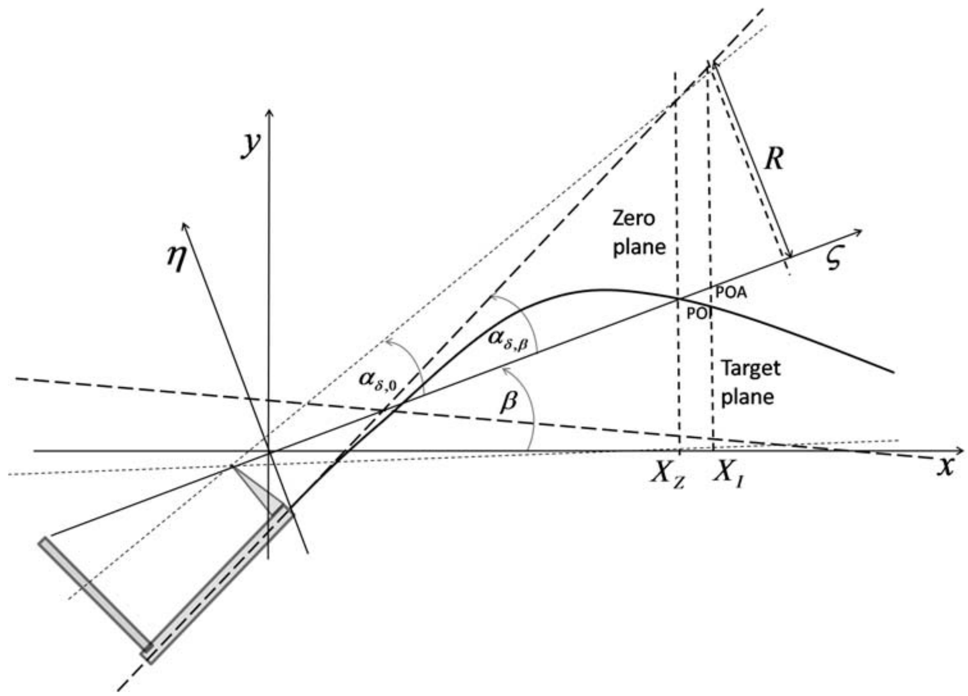
From Figure 7 you can extract R from the relationship $R = -\delta + \alpha_{\delta,\beta}\left(\frac{X_I}{\cos\beta} + R\tan\beta\right)$,

$$R = \left(-\delta + \alpha_{\delta,\beta}\frac{X_I}{\cos\beta}\right)\frac{1}{1 - \alpha_{\delta,\beta}\tan\beta} \qquad \text{(EQ 15)}$$

---

1. If you impart a full rotation of the gun about the line of sight, the POI describes a circle with parametric equations $\Delta Y = -\delta\left(\frac{X_I}{X_Z} - 1\right)(1 - \cos\varphi) - \alpha_{0,0}X_I(1 - \cos\varphi)$, $\Delta Z = \delta\left(\frac{X_I}{X_Z} - 1\right)\sin\varphi + \alpha_{0,0}X_I\sin\varphi$.

FIGURE 7. Geometric argument in the case of uphill shooting.



Replacing $\alpha_{\delta,\,\beta} = \dfrac{\delta}{X_Z}\cos\beta + \alpha_{0,\,\beta}$,

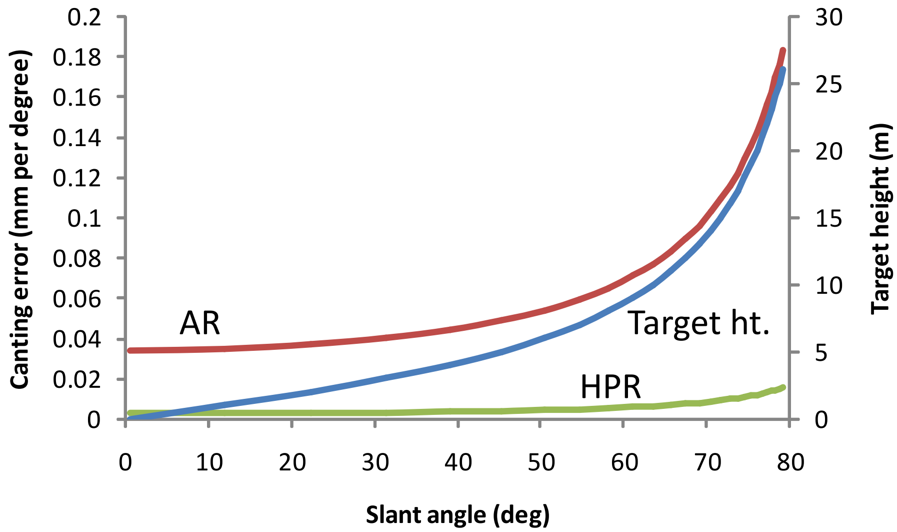
$$R = \left[\delta\left(\frac{X_I}{X_Z} - 1\right) + \alpha_{0,\,\beta}\frac{X_I}{\cos\beta}\right]\frac{1}{1 - \alpha_{0,\,\beta}\tan\beta - (\delta/X_Z)\sin\beta}, \qquad \text{(EQ 16)}$$

where $\alpha_{0,\,\beta}$ is given by EQ 6. Remember from 3.1 that $4\tan\beta\alpha_{0,\,0} \ll 1$ was a necessary condition for applicability of the Rifleman's Rule when the sight height is zero. Since under the Rifleman's Rule $\alpha_{0,\,\beta} \cong \alpha_{0,\,0}$, the denominator in EQ 16 says you can still apply the Rifleman's Rule when the sight height is not zero if, in addition to $4\tan\beta\alpha_{0,\,0} \ll 1$, $\delta/X_Z \ll 1$, which is almost always the case.

Notice that in this derivation there is a simplified treatment of the origin of coordinates, which causes a small first-order dependence of cant sensitivity on sight height to persist even when the impact distance and the zero distance coincide.

Figure 8 shows cant sensitivities in mm per degree of cant for a typical Field Target competition airgun and a high power rifle firing at a target at a horizontal distance of 5 m and varying height. The sight heights are not specified because the results are insensitive to the sight height. Notice that, under these conditions, the airgun is much more sensitive to cant than the high power rifle. As the angle of slant approaches 80 degrees, the cant sensitivity of both guns rises quickly. Figure 8 shows results for uphill firing. For downhill firing, the sensitivity curves are almost perfectly symmetrical and the target height curve is perfectly anti-symmetrical.

FIGURE 8. Sensitivities versus angle of slant for an Air Rifle (AR) and a High Power Rifle (HPR), firing from horizontal distance of 5 m with exit velocities 250 m/s and 850 m/s, respectively



4.3 Vacuum versus full trajectory cant sensitivities

To compute the cant sensitivity you need an estimation of either the elevation or the bullet drop. If you know the elevation that zeroes your gun at a given distance, you can use that in the formulas we developed earlier. If you know the bullet drop at a given distance for a projectile fired horizontally, you can estimate the elevation angle for zero sight height,

$$\alpha_{0,\,0} = \frac{\text{Drop at } X_Z}{X_Z} - \frac{\delta}{X_Z} \qquad \text{(EQ 17)}$$

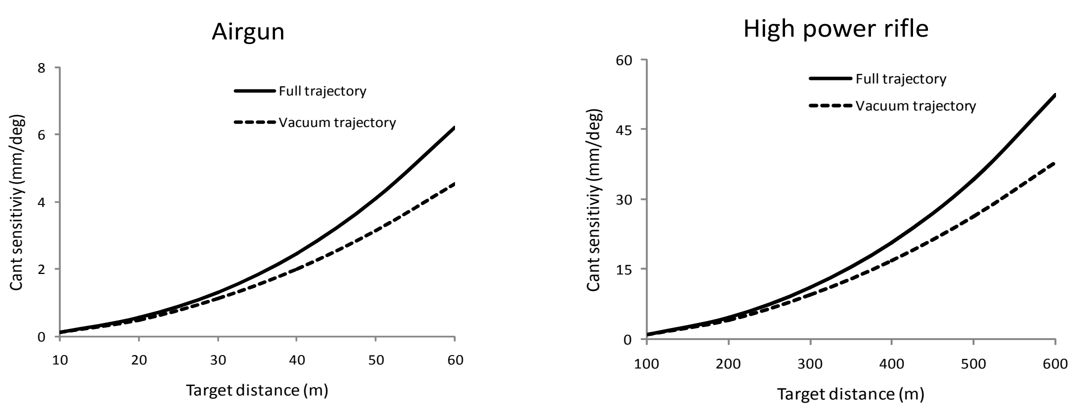
and use this value in the formulas developed here to get a very accurate calculation of the of the cant sensitivity. If you don’t have an estimation of either the elevation or the bullet drop, you can use the vacuum trajectory as a proxy.

To see to what extent the simple analytical formulation of the vacuum trajectory can illuminate what happens with projectiles under drag, let’s compare the cant sensitivities and elevations for two typical cases, a 0.648 g (10 gr) pellet fired at sea level with exit velocity 260 m/s and a high-power 14.25 g (220 gr) bullet fired at sea level with exit velocity 900 m/s. In both cases, the sight height is 40 mm. These are typical values of modern competition field target air rifles and high-power rifles.

The panels in Figure 9 show the one-degree cant sensitivities in the range 10 to 60 meters for an air rifle and 100 to 600 meters for a high-power rifle with the projectiles and muzzle velocities described in the previous paragraph. The G1-ballistic coefficients of the bullet and the pellet are 0.5 and 0.02, respectively. Remember that the ballistic coefficient is not a unit-independent quantity. Although the results are in metric units, the BC is in $\mathrm{lb/in^2}$, as is standard practice.

In both cases the vacuum trajectory reflects about two thirds of the cant sensitivity in the longer distances, and significantly better in the shorter distances. Notice the magnitudes involved. At 600 meters the cant sensitivity of a high-power rifle is of the order of 50 mm per degree. Assuming a cant uncertainty of +/- 1 degree, this results in a total uncertainty of about 100 mm. This is significant since the size of the X-ring in a 600 yd target in military target competitions has a diameter of 305 mm (12 in). In the air rifle case the situation is similar. At 1,000 m the +/- 1 degree sensitivity of a high power rifle is of the order of 400 mm, which is larger than the size of a human head. This is extremely relevant to a military sniper. At distances greater than 40 m, targets in Field Target competition have a diameter (so-called kill-zone) of 40 mm. This means that at 55 m, the +/- one degree cant sensitivity would account for about one fourth of the kill zone diameter. These observations suggest a competitor or a military sniper must be able to control his o her cant with a precision better than a degree.

FIGURE 9. Airgun and high-power rifle cant sensitivity as a function of target distance, in mm per degree of cant.



5.0 Summary

For convenience, here is a summary of the formulas you need to calculate cant sensitivity. X_Z and X_I are the horizontal projections of your zero distance and impact distance, respectively.

Elevation angle at zero sight height and zero slant, $\alpha_{0,0}$

1. From drop tables, your range measurement firing horizontally, or from ballistic program output setting slant to zero,

$$\alpha_{0,0} = \frac{\text{Drop at } X_Z}{X_Z} - \frac{\delta}{X_Z} \qquad \text{(EQ 18)}$$

2. From vacuum trajectory proxy,

$$\alpha_{0,0} = \frac{1}{2} g \frac{X_Z}{V^2} \quad \text{(EQ 19)}$$

Elevation angle with sight height δ and zero slant, $\alpha_{\delta, 0}$

1. From drop tables, range measurement or ballistic programs,

$$\alpha_{\delta, 0} = \frac{\text{Drop at } X_Z}{X_Z} \quad \text{(EQ 20)}$$

2. From ballistic programs with zero at X_Z and sight height δ,

$$\alpha_{\delta, 0} = \alpha_{0, 0} + \frac{\delta}{X_Z} \quad \text{(EQ 21)}$$

Elevation angle at zero sight height and at slant β, $\alpha_{0, \beta}$

1. If $4 \tan\beta \alpha_{0, 0}$ is small compared with 1 (this is almost always the case),

$$\alpha_{0, \beta} = \alpha_{0, 0} \quad \text{(EQ 22)}$$

2. If $4 \tan\beta \alpha_{0, 0}$ is not small compared with 1, or if you want a more accurate calculation at fairly large slant angle,

$$\alpha_{0, \beta} = \frac{1}{2} \cot\beta [1 - \sqrt{1 - 4\tan\beta \alpha_{0, 0}}] \quad \text{(EQ 23)}$$

Elevation angle at sight height δ and inclination β, $\alpha_{\delta, \beta}$

$$\alpha_{\delta, \beta} = \alpha_{0, \beta} + \frac{\delta}{X_Z} \cos\beta \quad \text{(EQ 24)}$$

Cant error in meters per radian of cant - all distances in meters

1. Horizontal firing,

$$R = \delta\left(\frac{X_I}{X_Z} - 1\right) + \alpha_{0, 0} X_I \quad \text{(EQ 25)}$$

2. Uphill or downhill firing with arbitrary slant and arbitrary sight height,

$$R = \left[\delta\left(\frac{X_I}{X_Z} - 1\right) + \alpha_{0,\beta}\frac{X_I}{\cos\beta}\right]\frac{1}{1 - \alpha_{0,\beta}\tan\beta - (\delta / X_Z)\sin\beta} \qquad \text{(EQ 26)}$$

Cant error in mm per degree of cant - R in meters**,**

$$\eta = 5.555\pi R \qquad \text{(EQ 27)}$$

6.0 Conclusions

The sensitivity of the point of impact to cant rotation is, for practical purposes, independent of the sight height as long as the gun is zeroed at the target distance. This is true both for horizontal and slanted shooting. What underlies this independence is the assumption that the effect of gravity is to cause the projectile to drop by the same amount over the distance to the target, *regardless of trajectory orientation and shape*. Under this assumption, in horizontal shooting, a 360-degree rotation of the gun about its line of sight would cause the gun to print points of impact on a circle of a radius that depends only on the elevation angle. In the case of slant shooting, the gun would print impact points on an ellipse whose shape will depend both on the elevation angle and the slant angle. Although this assumption of independence is an extremely good one, it is not strictly true, since the effect of gravity is also to distort the circle (or ellipse) printed by the points of impact. This distortion is, however, too small to be of practical significance.

Appendix

Estimating the sight height

In the case of iron sights, you can measure the sight height directly and without ambiguity as the distance between the center of the bore a the exit plane and the top of the front sight. In the case of a telescopic sight, the sight height is not directly observable. One reason for this is that the scope sits at some distance from the muzzle, making a direct measurement with respect to the bore axis at the muzzle impossible. Another reason is that the external geometry of the scope and its mounts, which shooters commonly use to estimate the sight height, is a proxy for what is an optical problem. In applications where you need precise knowledge of the trajectory to establish a zero, such as firing at short ranges with a precision airgun, accurate trajectory calculations require accurate estimates of the scope height. In cases like these it may be appropriate to regard the sight height as an *implied parameter*, rather than an observable quantity.

If you know the drag coefficient (or the Ballistic Coefficient (BC), its dimensional equivalent), a practical procedure to imply the sight height is to establish, by means of numerically computed trajectories, a theoretical zero at some distance using a crudely-estimated scope height, and then infer the sight height that matches the empirically observed vertical position of the POI at a much shorter distance by correcting the earlier estimation of the scope height. This iterative procedure works because the sight height has a much greater influence on the POI at shorter distances than at longer distances.

The vacuum trajectory affords a simple and effective version of this procedure. I'll illustrate the procedure with horizontal firing. The vacuum trajectory is a parabola,

$$y = -\frac{\alpha_{0,0}}{X_Z}x^2 + \left(\alpha_{0,0} + \frac{\delta}{X_Z}\right)x - \delta \qquad \text{(EQ 28)}$$

In this formula the gun is zeroed at X_Z and $\alpha_{0,0}$ is the elevation angle that causes the gun to be zeroed at X_Z when $\delta = 0$ and $\beta = 0$. By evaluating the trajectory at two locations with POI coordinates X_1, Y_1 and X_2, Y_2, respectively, you get an explicit equation for the sight height,

$$\delta = X_Z\left(\frac{Y_1 - Y_2}{X_1 - X_2} - \alpha_{0,0}\right) + \alpha_{0,0}(X_1 + X_2) \qquad \text{(EQ 29)}$$

Choosing $X_1 = X_Z$, and $X_2 = X_{\text{near}}$, where X_{near} is a distance much closer to the muzzle than X_Z,

$$\delta = \alpha_{0,0}X_{\text{near}} - \frac{X_Z Y_{\text{near}}}{X_Z - X_{\text{near}}} \qquad \text{(EQ 30)}$$

Replacing $\alpha_{0,0}$ from EQ 3,

$$\delta = X_Z\left(\frac{1}{2}g\frac{X_{\text{near}}}{V^2} - \frac{Y_{\text{near}}}{X_Z - X_{\text{near}}}\right) \qquad \text{(EQ 31)}$$

where g is the acceleration of gravity and V is the muzzle velocity. This equation allows you to accurately compute the sight height from observed impact points, without having to measure the scope height directly. Notice that this equation is independent of the ballistic properties of the projectile. For an airgun, a good choice of parameters is $X_Z = 22$ m and X_{near} as close to the muzzle as your optics will allow (usually 8 to10 meters.)