<u>Evaluation of Slow-Mode Damping Factor</u>

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James A. Boatright

In linear aeroballistics, the kinetic energy loss due to yaw-drag ΔE_α at a coning angle-of-attack α over the period T_2 of each “slow-mode” coning cycle can be written as

$$\Delta E_\alpha(T_2) = q*S*V*\alpha^2 *CD_\alpha*T_2.$$

We can also formulate the instantaneous kinetic energy E_C of the orbital coning motion itself as

$$\begin{aligned} E_C &= (m/2)*(r*\omega_2)^2 \\ &= (m/2)*(D*\text{Sin}\alpha*\omega_2)^2 \\ &= (m/2)*(D*\omega_2)^2 *\alpha^2 \end{aligned}$$

where $r = D*\text{Sin}\alpha \approx D*\alpha$ is the coning radius of the CG of the bullet orbiting around a “mean CG” location moving smoothly along the “mean trajectory” of the bullet at its “mean velocity,” and **D** is the slowly varying coning distance of the CG of the bullet from its coning apex, each given in **feet**.

Now, as the coning angle α decreases (due to frictional damping for any ***dynamically stable*** projectile) from its initial value α_0 to its final value α_1 at the completion of this coning cycle, the change ΔE_C in orbital coning energy can be written as

$$\Delta E_C = (m/2)*(D*\omega_2)^2 *(\alpha_0^2 – \alpha_1^2)$$

$$\Delta E_C = m*(D*\omega_2)^2 *[(\alpha_0 + \alpha_1)/2]*(\alpha_0 – \alpha_1)$$

$$\Delta E_C = m*(D*\omega_2)^2 *\alpha*\Delta\alpha$$

where $(\alpha_0 + \alpha_1)/2 = \alpha$, the *average* coning angle over this cycle,

and $\alpha_0 – \alpha_1 = \Delta\alpha > 0$, the *reduction* in coning angle due to damping.

We now hypothesize that, at least in **hyper-stable flight** in which no nutation needs damping, and for **dynamically stable bullets**, the average loss in “forward motion” kinetic energy ΔE_α over any coning cycle due to flying with an *average* aerodynamic angle-of-attack α ***causes*** the *average* "frictional damping" decrease in orbital coning energy ΔE_C during that same coning cycle. That is to say, we are tentatively assuming that a small fraction **e** ($e \approx 0.0025$) of the yaw-drag of the dynamically stable rifle bullet directly causes the damping of its coning angle α in *steady-state, minimum coning angle, "hyper-stable" flight*. Then, these two energy losses must be proportionally related to each other.

If this hypothesis is true, we can set $\Delta E_C = e*\Delta E_\alpha$ over any particular coning cycle, where the constant fraction **e** is greater than **zero** for a ***dynamically stable*** projectile, but not greater than **1.0**, and so that

$$m*(D*\omega_2)^2 *\alpha*\Delta\alpha = q*S*V *\alpha^2 *e*CD_\alpha*T_2$$

or, dividing through by $\alpha^2 > 0$ and by $[m*(D*\omega_2)^2]$,

$$(\Delta\alpha)/\alpha = (T_2)*[q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2].$$

We recognize this expression as having the form of the classic exponential damping of the coning angle α:

$$\alpha(t) = \alpha_0*\exp[-\lambda_2*t]$$

with $$\lambda_2 = [q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2].$$

If we replace the coning period T_2 with a small increment in time **dt**, and replace $\Delta\alpha$ per coning cycle with a small decrement $-d\alpha$ in α, then in the limit as **dt** approaches zero, this expression becomes

$$d\alpha/\alpha = -\lambda_2*dt$$

After integrating both sides from **0** to **t**,

$$\ln[\alpha(t)/\alpha(0)] = - \lambda_2*t$$

Or, after exponentiating

$$\alpha(t) = \alpha_0*exp[-\lambda_2*t] \qquad [QED].$$

Thus, we have derived the long-accepted aeroballistic exponential damping relationship from the basic physics of our hypothesis that a portion **e** of the yaw-drag kinetic energy loss causes the damping of the coning angle for dynamically stable bullets in hyper-stable flight. [In "hyper-stable" flight in flat-firing, the spin-stabilized projectile is flying with an aerodynamic angle-of-attack **α** equal to the change in flight path angle **Φ** caused only by gravity during each half coning cycle. This coning angle **α** is typically less than **0.1 degrees**, and produces negligible yaw-drag.]

If only a small fraction **e ($0 < e \leq 1.0$)** of this yaw-drag induced kinetic energy loss is actually responsible for frictional damping of the coning angle **$\alpha(t)$** for a dynamically stable bullet, we accommodate that simply by using **$e*CD_\alpha$** in the above expression for **λ_2**:

$$\lambda_2 = [q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2].$$

The ***initial coning rate* ω_2** has been shown[1] always to be ***directly proportional*** to the selected rifling twist rate **n** given in calibers per turn, and none of the other parameters in this expression depend upon the selected rifling twist-rate **n**. [While **n** is used in evaluating **e** from data measurements for any given bullet, **e** is subsequently considered as a bullet constant.]

Therefore, the slow-mode damping factor **λ_2** as formulated above varies ***inversely with the square of*** **n**. Smaller values of **n** produce larger damping factors **λ_2** and quicker "damping out" of any initial coning angle **α_0** and, thus, produce significantly less total kinetic energy loss over a long flight due to any given ***initial yaw attitude*** **α_0**.

It is worth pointing out here that if **$\lambda_2 = f_2$**, the coning rate in hertz, the coning angle **$\alpha(t)$** will be ***critically damped*** and reduced by a factor of **0.368** during each early coning cycle.

The total forward velocity **ΔV** lost by the rifle bullet due to yaw-drag while any initial coning angle **α_0** is being damped out to ***insignificance*** can be well approximated from

$$\Delta KE = (m/2)*(\Delta V)^2$$

$$\Delta KE = q*S*CD_\alpha*V_0*\alpha_0^2 \int exp[-2*\lambda_2*t]dt$$

$$\Delta KE = q*S*CD_\alpha*V_0*\alpha_0^2/(2*\lambda_2).$$

Here, we are integrating the drag-force component due to yaw in linear aeroballistics from **t = 0** at the beginning of ballistic flight to some arbitrary later time **T** when minimum coning angle "hyper-stable" flight has been achieved. For a dynamically stable rifle bullet fired from a very quick-twist (**$n \approx 20$**) barrel, **T** is just a very few early coning periods **T_2**, and use of ***initial values*** is justified in this approximation.

From this expression, the ***total retardation*** of the bullet over a long-distance flight **ΔV** due to early yaw-drag is ***directly proportional to the first power of* n**, the rifling twist-rate in calibers per turn. This yaw-drag velocity loss occurs very early in ballistic flight and is never regained in flat firing; so, it *reduces* the **Ballistic Coefficient (BC)** measured for each bullet depending upon its particular initial non-zero yaw disturbance **α_0**, which is consistent with our test-firing results.

When evaluating the coefficient of drag for exactly nose-forward flight **CD_0** for any spin-stabilized projectile from measurements of a series of shots, only those shots having the ***smallest*** times-of-flight to the target, after correcting for any variation in launch speeds, are the most truly representative. Variable yaw-drag, caused by even small random yaw destabilization during launch, can only increase measured times-of-flight.

Explanatory Note 1:

The initial spin-rate **f** of the fired bullet in hertz is equal to the muzzle speed **V_0** of the bullet in feet per second divided by the twist-rate (**n*d** in feet) of the rifling in the firing barrel.

From the Tri-Cyclic Theory, the initial coning rate **ω_2** of the bullet in radians per second is then given by

$$\omega_2 = 2\pi*(Ix/Iy)*[V_0/(n*d)]/(R + 1)$$

where

Ix, Iy = Second moments of inertia of the bullet about its crossed principal axes

n = Twist-rate in calibers per turn

d = Caliber in feet

$R = \omega_1/\omega_2 = f_1/f_2$ = Gyroscopic stability ratio, such that

$Sg = (R + 1)^2/(4*R)$ = Gyroscopic stability.

According to information from a 1989 paper by Beat P. Kneubuehl of Thun, Switzerland, the initial value of (**R + 1**) can be estimated for a simple cone-on-cylinder projectile model as

$$\mathbf{R + 1 = (Ix/Iy)*\{[32\pi*Ix]/[n^2 *\rho*d^5 *CM_\alpha]\}}$$

where $\boldsymbol{\rho}$ **=** Air density in slugs per cubic foot

$\mathbf{CM_\alpha}$ **=** Overturning moment coefficient in aeroballistics.

Both **Sg** and **R** vary ***inversely with the square of twist-rate*** **n**.

This is consistent with the classic aeroballistic definition of $\mathbf{Sg = P^2/(4*M)}$ in terms of the auxiliary parameters **P** and **M**. **P** is a generalized spin-rate parameter (varying with $\mathbf{n^{-1}}$) and **M** is a generalized overturning moment parameter as they are used in aeroballistics.

So, the initial coning rate $\boldsymbol{\omega_2}$ is ***directly proportional to the selected twist-rate*** **n** for the rifling.

For example, increasing **n** from **20** to **24 calibers per turn** (i.e., slowing the rifling twist-rate by **20-percent**) would increase the initial coning rate $\boldsymbol{\omega_2}$ of the rifle bullet by **20-percent**.

Importantly, this slowing of the twist-rate of the rifling by **20-percent** would simultaneously

(1) ***increase*** the total retardations $\mathbf{\Delta V}$ caused by any random initial yaw disturbances $\boldsymbol{\alpha_0}$ by **20-percent** by decreasing the value of the slow-mode damping factor $\boldsymbol{\lambda_2}$ by **44-percent**, and

(2) ***decrease*** the shot-group spread on the target by **20-percent** caused by aerodynamic jump deflections of the trajectory due to these same randomly oriented initial yaw disturbances, by reducing by **20-percent** the integration time $\mathbf{T_2/2}$ over which the cross-track impulse is accumulated.