Rifle Bullet Stability

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General

These thoughts the stability of rifle bullets in flight apply primarily during their supersonic flights in flat firing to long ranges. While rifle bullet stability in the muzzle-blast zone and far downrange in the transonic and subsonic flight regimes are not yet fully investigated, a few comments and observations will be included about those rifle bullet stability concerns.

Gyroscopic Stability

Gyroscopic Stability (**Sg**) refers to the ability of a spin-stabilized, *statically unstable* rifle bullet to resist tumbling in flight caused by its aerodynamic overturning moment. Basically, **Sg** is formulated as the ratio of the spinning bullet's *axial rigidity* to its *aerodynamic overturning moment* at any point during its ballistic flight.

A rifle bullet is termed *statically unstable* because its center of mass is almost always located behind its center of aerodynamic pressure **CP** in aeroballistic flight. Herein, we shall use the more common term "center of gravity" **CG** interchangeably with the more proper term "center of mass." With its **CP** located ahead of its **CG**, a rifle bullet always tends to swap ends in aeroballistic free flight. A high initial rate of spin is imparted to the fired bullet by the rifling of the barrel to prevent its erratic tumbling in aeroballistic flight.

The *axial rigidity* of the free flying bullet is solely attributable to its angular momentum **L** about its spin axis. **L** is a vector quantity pointing forward along the spin-axis of a right-hand spinning rifle bullet, which is the only spin direction being considered here to avoid unnecessary complexity.

The *aerodynamic overturning moment* **M** causes the spinning bullet, acting as a free-flying gyroscope in ballistic flight, to *precess* and *nutate* in flight instead of tumbling erratically. These two gyroscopic reactions cause the spin-axis of the bullet to trace out its familiar epicyclic motions in "aircraft type” pitch and yaw attitude angles. As a rotationally symmetric solid of revolution, the aeroballistic angle-of-attack **α** of a rifle bullet is the root-sum-squares (RSS) of its aircraft-type pitch and yaw attitude angles. This angle-of-attack **α** is conventionally termed a “yaw” attitude in aeroballistics work.

The slow-mode motion of the spin-axis direction is *gyroscopic precession*, and the fast-mode motion is *gyroscopic nutation*. This slow-mode motion is also called “coning motion” in Coning Theory. When the spinning rifle bullet is flying with a non-zero aeroballistic angle-of-attack **α**, the overturning moment **M** (a torque vector) rotates in roll orientation along with the slow-mode gyroscopic precession of the spin-axis of that bullet.

The ***angular rates*** of these fast-mode and slow-mode spin-axis epicyclic motions are **ω_1** and **ω_2**, respectively, in **radians per second**. They are readily found from the Tri-Cyclic Theory relationships:

$$\omega_1 + \omega_2 = (Ix/Iy)*\omega = \omega_2*(R + 1)$$
$$R = \omega_1/\omega_2$$
$$Sg = (R + 1)^2/(4*R)$$

where **ω** is the instantaneous spin-rate of the rifle bullet itself in **radians per second** and **Ix/Iy** is the dimensionless ratio of the second moments of inertia about crossed principal axes for the mass distribution of that bullet.

The spin-rate of the bullet **ω** decreases very nearly exponentially with time of flight **t**:

$$\omega(t) \approx \omega(0)*\exp[(-0.0321/(12*d))*t]$$

where **d = Caliber of bullet in feet**.

The stability ratio **R** is a better, more sensitive, indicator of gyroscopic stability than is **Sg** itself.

$$R = 2*\{Sg + SQRT[Sg*(Sg - 1)]\} - 1.$$

Sg is classically defined in aeroballistics as

$$Sg = P^2/(4*M) = (\omega_1 + \omega_2)^2/(4*\omega_1*\omega_2) = (R + 1)^2/(4*R).$$

In particular, we often need to find the *initial* gyroscopic stability **Sg** of a just-fired bullet. We can find it from the initial stability ratio **R**.

$$R + 1 = (Ix/Iy)*[\omega(0)/\omega_2(0)]$$

Substituting into the above expression for the initial value of **R + 1**, and simplifying, we have

$$R + 1 = (Ix/Iy)*\{[2\pi*V/(n*d)]^2 *(m*d^2 *kx^2)/[(\rho/2)*V^2 *(\pi/4)*d^3 *CM_\alpha]\}$$

$$R + 1 = (Ix/Iy)*\{[32\pi*m*kx^2]/[n^2 *\rho*d^3 *CM_\alpha]\}$$

For the monolithic copper Mark II ULD bullets of any caliber **d** in **inches**,

$$Iy/Ix = 14.0$$
$$kx^{-2} = 9.0$$
$$\rho_P = 8.84 \text{ gm/cc} = 2235.6 \text{ grains/in}^3$$
$$\text{Volume of bullet} = 3.15*d^3$$
$$m = \rho_P*\text{Volume of bullet}$$
$$\rho_P/\rho = [12^3 *2235.6/7000]/0.0764742 \text{ lbm/ft}^3 = 7216.5 \text{ (ICAO)}$$
$$n = 20 \text{ calibers/turn}$$
$$R + 1 = 10.97$$
$$R + 1 = (Ix/Iy)*\{[32\pi*3.15*(\rho_P/\rho)*kx^2]/[n^2 *CM_\alpha]\}$$

So, for a sea-level ICAO standard atmosphere

$$CM_\alpha = [32\pi*3.15/(14.0*9.0)]*[(7216.5)/20.0^2]/10.97 = 4.1333$$

And,

$$R(0) = (4179/n^2) - 1.$$

As **n** varies by about a factor of **2** for target rifles, from about **20 calibers per turn** down to about **40 calibers per turn** in twist-rate, **R(0)** varies by about a factor of **4** for the same bullet fired from different target rifles.

Then,

$$\text{Initial Sg} = [R(0) + 1]^2/[4*R(0)]$$
$$\text{Initial Sg} \approx (4366110/n^4)/[(4179/n^2) - 1] .$$

Note that the initial gyroscopic stability **Sg** and its stability ratio analog **R** are independent of the caliber **d** among rifle bullets of the same design and construction.

We shall further explore the topic of gyroscopic stability below.

Tri-Cyclic Rates of Epicyclic Motion

Right out of the muzzle, the initial spin-rate $\boldsymbol{\omega_0}$ of the rifle bullet is

$$\boldsymbol{\omega_0 = 2\pi * V(0)/(n*d)}$$

where **n = Barrel twist-rate in calibers/turn**
d = Caliber in feet.

As mentioned earlier, the spin-rate of the bullet $\boldsymbol{\omega(t)}$ slows only gradually with ongoing time-of-flight **t**:

$$\boldsymbol{\omega(t) \approx \omega_0 * \exp[(-0.0321/(12*d))*t]}$$

where **d = Caliber of bullet in feet**.

We also know, as derived in Coning Theory, that

$$\boldsymbol{\omega_2(t) = q*S*d*CM_\alpha/(Ix*\omega)}$$

where $\boldsymbol{q = (\rho/2)*V^2}$
$\boldsymbol{S = (\pi/4)*d^2}$
$\boldsymbol{Ix = m*d^2 *kx^2}$
d = Caliber in feet.

From the physics of gyroscopes, we have the vector cross-product relationship for the precession rate vector $\boldsymbol{\omega_2}$ (pointing along the axis of the bullet's coning motion) in terms of the angular momentum vector **L** of the spinning gyroscope (bullet) and the aerodynamic overturning moment vector **M** acting upon it:

$$\boldsymbol{\omega_2 \times L = M}$$

This vector relationship is derived for a "fast, heavy top" in classical mechanics. Any spin-stabilized rifle bullet mechanically qualifies as this type of gyroscope.

From Coning Theory, we know that the coning axis, and thus the precession vector $\boldsymbol{\omega_2}$, points forward along the neutral torque axis into the direction of approach of the apparent wind as seen by the flying bullet. For non-zero coning angles $\boldsymbol{\alpha}$, the angle between the vectors $\boldsymbol{\omega_2}$ and **L** is just the coning angle-of-attack **α**. The angular momentum vector **L** points forward along the spin-axis of a right-hand spinning bullet. Angular momentum is ***conserved*** in physics and cannot undergo any step-changes in magnitude or direction. Variation in **L(t)** must be ***continuous*** in time **t**.

In ***magnitudes***, this vector cross-product relationship becomes

$$\omega_2 * L * \mathrm{Sin}(\alpha) = M$$

In aeroballistics, the overturning moment **M** acting upon the flying bullet is given as

$$M = q * S * \mathrm{Sin}(\alpha) * d * CM_\alpha$$

and, for small, steady angles-of-attack **α**,

$$\partial M / \partial \alpha = q * S * d * CM_\alpha$$

Therefore, for non-zero angles of attack **α**, the magnitude of the coning rate ω_2 can be written as

$$\omega_2 = q * S * d * CM_\alpha / L$$

or

$$\omega_2 * L = q * S * d * CM_\alpha$$

$$\omega_2 * L = \partial M / \partial \alpha.$$

Of course, the angular momentum **L** of any real spin-stabilized rifle bullet must also be always non-zero.

The direct derivation of this expression from aeroballistics and the physics of a fast, heavy top constitutes another validation of Coning Theory.

Since we have from Tri-Cyclic Theory,

$$L = I_X * \omega = I_Y * (\omega_1 + \omega_2) = I_Y * (R + 1) * \omega_2$$

we can also say,

$$(\omega_2)^2 = q * S * d * CM_\alpha / [I_Y * (R + 1)].$$

This is a useful expression for finding the slow-mode coning rate $\boldsymbol{\omega_2(t)}$ at any time **t** during the flight of a spin-stabilized projectile based on its aeroballistic parameters. Only **q**, $\mathbf{CM_\alpha}$, and **R** vary with time-of-flight **t** for most projectiles.

Note that the slow-mode coning rate $\boldsymbol{\omega_2(t)}$ is essentially completely independent of the coning angle $\boldsymbol{\alpha}$ as postulated in Coning Theory, even though for large angles-of-attack $\boldsymbol{\alpha}$, $\mathbf{CM_\alpha}$ does gradually roll-off to smaller values. Coning motion is an example of *simple harmonic motion*, and as such the *amplitude* and *frequency* of that motion are mutually independent variables.

If we know the stability ratio **R(t)** at any time during the flight we can find the fast-mode nutation rate $\boldsymbol{\omega_1}$ from

$$\boldsymbol{\omega_1 = R^*\omega_2}.$$

Otherwise, we can evaluate the nutation rate $\boldsymbol{\omega_1}$ at any time during the flight from the Tri-Cyclic relationship

$$\boldsymbol{\omega_1 = (I_X/I_Y)^*\omega - \omega_2}.$$

Both cyclic rates, $\boldsymbol{\omega_1}$ and $\boldsymbol{\omega_2}$, slow monotonically with ongoing time-of-flight **t** in flat firing because the spin-rate $\boldsymbol{\omega(t)}$ gradually slows monotonically and the stability ratio $\mathbf{R = \boldsymbol{\omega_1/\omega_2}}$ only very gradually increases in supersonic flight, monotonically as well.

The stability ratio **R** and gyroscopic stability factor **Sg** increase during supersonic flight because the spin-rate $\boldsymbol{\omega(t)}$ of the rifle bullet decays much more slowly than does the square of that bullet's forward velocity $\mathbf{V(t)^2}$, which aerodynamically determines its dynamic pressure **q**. The aerodynamic forces of drag and lift and the aerodynamic overturning moment are each proportional to the instantaneous dynamic pressure **q** as they are modelled in aeroballistics.

Bullet Destabilization in the Muzzle-Blast Zone

From the time the rifle bullet exits the crown of the muzzle until it commences aerodynamic ballistic flight several yards downrange after penetrating the muzzle-blast shockwave, it is subjected to destabilization in aeroballistic yaw $\boldsymbol{\alpha}$ and yaw rate **$d\alpha/dt$**. Even if the bullet left the muzzle of the rifle barrel perfectly, with **$\alpha = d\alpha/dt = 0$**, it can still acquire significantly non-zero yaw attitude and yaw-rate while transiting this muzzle-blast zone. Any non-zero initial aeroballistic yaw or yaw rate will then cause an ***aerodynamic jump*** during the first half coning cycle occurring in subsequent ballistic flight. These randomly oriented angular jump deflections eventually affect bullet impact points on any downrange target by reducing bullet placement accuracy and increasing shot-group sizes. The destabilized rifle bullet also suffers increased yaw-drag until its initial coning angle-of-attack damps out.

Heavy conventional lead-cored, jacketed rifle bullets have long minimized this destabilization within the muzzle-blast zone by having relatively higher angular momentum **L** at launch and by incorporating convex bases and large-radius rear corners into their traditional designs. For a given caliber **d** of match-type rifle bullet, the radius of gyration about its spin-axis (**k_X**) tends to be similar regardless of bullet design and construction. Monolithic copper-alloy bullets have only about **80 percent** of the average material density **ρ_P** of similar lead-cored bullets. Therefore, they must be spun at least **25 percent** faster to enjoy a similarly high level of angular momentum **L** right out of the muzzle of the rifle barrel.

As will be shown in the next paragraph, the initial angular momentum **L** of a fired rifle bullet can be expressed as

$$\mathbf{L = I_X*\omega_0 = 26.1156*\rho_P*d^4 *V_0/n.}$$

The needed higher initial spin-rate **ω_0** is normally accomplished by firing the lighter copper-alloy bullet at a higher muzzle velocity **V_0** and by reducing the number **n** of **calibers per turn** defining the twist-rate of the rifling.

It might bear mentioning here that a similar bullet destabilization problem occurs much later in conventional forward aerodynamic flight when the slowing rifle bullet enters the turbulent transonic flight regime far downrange as its slowing airspeed approaches **Mach 1.0**.

Enhancing the bullet's initial angular momentum **L** by increasing its initial spin-rate $\boldsymbol{\omega_0}$ pays dividends in rifle bullet stability during that subsequent transonic flight regime as well. The spin-rate $\boldsymbol{\omega}$ of the bullet slowly decays almost exponentially with time of flight **t**, so that

$$\boldsymbol{\omega(t) \approx \omega_0 * \exp[-(0.0321/d))*t]}$$

where the caliber **d** is given in **inches**.

After the rifle bullet initially clears the muzzle, it is flying aerodynamically backwards through hot powder gases and particulates escaping the muzzle and passing up the fired bullet at a relative speed of about **6000 fps –** $\boldsymbol{V_0}$, depending primarily upon the muzzle pressure behind the exiting bullet. The ground-speed $\boldsymbol{V_0}$ of the rifle bullet is actually increased by a few percent due to a forward-acting aerodynamic drag force during the short time **Δt** while it is traversing the muzzle-blast zone.

An associated reverse aerodynamic lift force produces an overturning torque impulse **ΔM** which destabilizes the free-flying bullet in aeroballistic yaw and yaw-rate by about **ArcTan[ΔM/L]** in yaw **(Δα)** and by about **ArcTan[ΔM/(L*Δt)]** in yaw-rate **(Δα/Δt)**. The radial orientation of the overturning torque impulse **ΔM** is essentially random.

Both the overturning moment **M** and its impulsive form **ΔM** are inherently proportional to the ***1st power*** of the bullet's caliber **d** as formulated in aeroballistics. The long ogive of the rifle bullet acts as an effective lift-minimizing afterbody design feature of that bullet while it is flying in reverse. However, the large diameter of the "blunt meplat" of the bullet base enhances its reversed coefficients of lift, drag, and overturning moment with the ***square*** of that effective meplat diameter, which is usually significant fraction **f** of the bullet's caliber **d**, or $\mathbf{(f*d)^2}$. [The meplat-diameter-squared dependence for aerodynamic drag comes directly from Bob McCoy's McDRAG estimation program. We are here projecting that $\mathbf{(f*d)^2}$ dependence for lift and overturning moment as well.]

Therefore, the magnitude of the yaw-destabilizing torque impulses **ΔM** vary with the **cube** of bullet diameter $\mathbf{d^3}$, and the resulting disturbances in yaw

$\Delta\alpha$ and in yaw-rate **$\Delta\alpha/\Delta t$** are each inversely proportion to the ***1^{st} power*** of the projectile caliber, or **1/d**.

Two bullet design approaches come to mind for minimizing the effective aerodynamic diameter **f*d** of the bullet base while it is flying backwards without causing problems during subsequent forward aerodynamic flight. The base diameter **f** is about **0.842 calibers** for many ULD and ELR rifle bullet designs.

One possible bullet design modification is to bevel slightly (but not radius) the rear corners of the boat-tailed bullet base. A bevel angle of **45 degrees** off-axis would still allow the turbulent boundary layer flowing down the boat-tail surface to *detach* cleanly in forward aerodynamic flight, while reducing the effective aerodynamic meplat diameter **f*d** in reversed flight. Radiusing this rear corner instead would tend to pull that turbulent boundary layer around the rear corner in forward flight before it eventually detached, leading to increased alternate shedding of vortices into the turbulent wake of the forward flying bullet, destabilizing it in ballistic flight and increasing its base drag.

Another possible bullet design approach for reducing the large effective base diameter **f*d** is to make the base of the rifle bullet ***convex*** with a generating radius of **0.65 calibers**. [This value comes from a 1966 NASA-Langley subsonic wind tunnel study (NASA TN D-3388) of blunt elliptical ogives. This radius value is the maximum *radius of curvature* at the center of the nose, on the axis of symmetry of the bullet, for the most blunt of elliptical nose shapes which does not increase measured subsonic aerodynamic drag above that of a full hemispherical nose shape, **r = 0.50*d**.]

Both design approaches could be used simultaneously to reduce further the overturning torque impulse **ΔM** during reversed flight through the muzzle-blast zone without interfering with its forward-flight aerodynamics.

It should be pointed out here that a well-designed, effective muzzle brake or suppressor can significantly reduce the muzzle-blast destabilization of rifle bullets fired through them. These attached muzzle devices accomplish this by bleeding off the muzzle pressure in a controlled fashion while preventing the rifle bullet's ever having to "fly backwards" aerodynamically through the escaping high-pressure powder gases. A series of close-fitting metallic baffles is used in either type of add-on muzzle device to prevent reversed aerodynamic flight of the rifle bullet from occurring while the muzzle pressure is being bled off. The baffle holes should be well centered on the bore and have no more than **0.010-inch** radial clearance around the groove-diameter portion of the bullets fired through them.

In practice, long-range LR and extreme long-range ELR target rifles in calibers 338 through 510 normally utilize muzzle brakes or suppressors to control felt recoil during sustained firing from the shoulder. If properly designed, these devices inherently control bullet destabilization in the muzzle-blast zone. Target rifles made in calibers 224 through 308 often do not utilize muzzle brakes or suppressors for recoil control. Target bullets made in these smaller calibers especially need the design modifications mentioned above to control bullet destabilization within the muzzle-blast zone because they are inherently more easily destabilized than their larger-caliber cousins.

In the absence of any muzzle attachment, these smaller-caliber target rifles suffer accuracy losses proportional to **(0.330/d)** due to aerodynamic jump in subsequent ballistic flight, as discussed below. Group sizes for 6.5 mm bullets, for example, should be expected to exceed those of similar 338-caliber bullets of any given design by a factor of **1.29** at any firing distance.

Effects of Caliber on Bullet Stability

We should point out the rifle bullet's significant ***caliber-dependence*** in the magnitude of the yaw destabilization **$\Delta\alpha$** to be expected in the muzzle-blast zone:

$$\boldsymbol{\Delta\alpha = \mathrm{Tan}^{-1}[\Delta M/L] \approx \Delta M/L}$$

As shown above, the destabilizing torque impulse **ΔM** increases in magnitude with the cube of bullet caliber **d^3**.

The *axial rigidity* of the flying bullet is determined by the magnitude of its angular momentum vector **L**. The value of **L** determines how much increase in coning angle **$\Delta\alpha$** is incurred for any given amount of *overturning moment* torque **M** integrated over the time **Δt** of its application; i.e., for any given yaw-destabilizing *torque impulse* **ΔM** applied to the bullet in flight.

There is significant caliber-dependence in the bullet's angular momentum **L**, in both its second moment **I_X** and in its initial spin-rate **ω_0** components:

$$\boldsymbol{L = I_X * \omega}$$

The magnitude of the bullet's second moment of inertia **I_X** about its principal longitudinal spin-axis (its **x**-axis) can be expressed as:

$$\boldsymbol{I_X = m * d^2 * k_X^2.}$$

The spin-axis of the bullet is a *principal axis* of rotation because it is the axis of *minimum possible* second moment of inertia for its mass distribution. Any axis through the center of mass of the bullet resulting in an *extremum* (either a maximum or a minimum) of its second moment is a principal axis of rotation.

This second moment of inertia **I_X** about the bullet's principal axis is often given in units of **grain-inch^2**. For a typical monolithic extreme long-range (ELR) rifle bullet having an average density **ρ_P** in **grains/cubic inch**, a volume of **3.15 cubic calibers d^3**, and with its reference diameter **d** *now given in* **inches**:

$$\boldsymbol{m = 3.15 * \rho_P * d^3} \qquad \text{(in grains).}$$

A **grain** is actually **1/7000 of a pound**, which is properly a unit of force (weight). It is common when working in imperial units to use *pounds* to quantify *mass*, as long as that convention is well understood. The imperial unit of mass is properly the **slug**, which weighs **32.174 pounds** in a gravitational field of **32.174 feet/second^2**. The mass of one **slug** is **14.5939 kilograms.**

Since the radius of gyration k_X of the mass distribution of the bullet about its spin-axis in **calibers** (here used as dimensionless canonical units) is always very nearly **0.3316 calibers** for long-range monolithic rifle bullets,

$$I_X = m*d^2 *k_X^2$$
$$I_X = 3.15*(0.3316)^2 *\rho_P*d^5$$
$$I_X = 0.34637*\rho_P*d^5$$

with **d =** Reference diameter **(caliber)** in **inches**.

Thus, all else being equal, I_X varies directly with the ***5th power*** of caliber d^5.

Just out of the muzzle, the initial spin-rate ω_0 of the bullet is given in **radians/second** by:

$$\omega_0 = 2\pi*V_0/Tw = 2\pi*V_0*12/(n*d).$$

As mentioned earlier, the subsequent spin-rate $\omega(t)$ of the bullet slows only gradually, almost exponentially with time-of-flight **t.**

So, the spin-rate $\omega(t)$ of the rifle bullet typically varies inversely with its caliber d^{-1}, even when fired with the *same initial velocity* V_0 and, as we showed above, with the *same initial gyroscopic stability* **Sg** as with similar bullets of other calibers.

Combining these caliber-dependence expressions, the initial magnitude of the angular momentum **L** of the rifle bullet is given by:

$$L = I_X* \omega_0 = 26.1156*\rho_P*d^4 *V_0/n.$$

So, the initial angular momentum **L** of the rifle bullet varies directly with the ***4th power*** of its diameter **d** whenever the bullet design, its material density, its muzzle velocity, and the rifling helix-angle **(180/n** in **degrees**) are each held fixed in studying caliber-effects in isolation.

Since $\mathbf{I_X}$ is normally invariant for a fired rifle bullet in ballistic flight, the magnitude of its in-flight angular momentum **L** varies directly with the gradually slowing spin-rate $\boldsymbol{\omega}$ of the bullet; i.e., also decreasing almost exponentially with time of flight **t**.

Differing calibers of rifle bullets of similar design and materials, fired at the same muzzle velocity $\mathbf{V_0}$ from rifled barrels having the same helix-angle (**180/n**, in **degrees**), will always have angular momentum **L** proportional to the ***4th power*** of their caliber $\mathbf{d^4}$ at corresponding points throughout their flights.

All else being equal, larger-caliber rifle bullets are inherently much more rigid axially in flight than smaller-caliber bullets of similar design.

Since the destabilizing torque impulse $\mathbf{\Delta M}$ expected in the muzzle-blast zone was earlier shown to increase with the ***cube*** of the bullet's caliber $\mathbf{d^3}$, the expected amount of yaw destabilization $\boldsymbol{\Delta\alpha} = \mathbf{Tan^{-1}[\Delta M/L]}$ in the muzzle-blast zone should vary ***inversely*** with caliber $\mathbf{d^{-1}}$. Initial aerodynamic drag increases are modeled as proportional to $\mathbf{(\Delta\alpha)^2}$ and that drag increase would then vary inversely with the square of caliber $\mathbf{d^{-2}}$.

As mentioned above, in the absence of any muzzle attachment effectively controlling muzzle-blast destabilization for large-caliber bullets, smaller-caliber target rifles suffer accuracy losses proportional to **(0.330/d)** due to aerodynamic jump in subsequent ballistic flight, as discussed below. Group sizes for 6.5 mm bullets, for example, should be expected to exceed those of similar 338-caliber bullets of any given design by a factor of **1.29** at any firing distance.

Initial Dynamic Stability Considerations

Dynamic stability (**Sd**) deals with the ***rates of change*** in the angular ***amplitudes*** of the coning and nutation motions of the spin-axis of the bullet in flight. The slow-mode "coning motion" of the CG of the spinning bullet around the mean trajectory is a type of ***damped harmonic motion***. If the coning angle α, for example, decreases in size with ongoing flight time **t**, that slow-mode motion is said to be "damped." If $\alpha(t)$ increases in size or remains constant with ongoing time-of-flight, it is termed an "undamped" motion. Ballisticians have long modeled this damping as exponential in downrange flight distance (**s**) measured in projectile calibers (**d**) of travel distance with fast-mode and slow-mode damping factors λ_F and λ_S, respectively, given in **inverse calibers**.

The ***rates***, ω_1 and ω_2, of the two gyroscopic motions of the bullet's spin-axis direction are fixed by the physics of gyroscopic motion, and they are not affected in any way by the dynamic damping mentioned above. However, each of the two gyroscopic rates, ω_1 and ω_2, decreases monotonically over time of flight **t** as the spin-rate of the bullet $\omega(t)$ in **radians per second** slows (very nearly) exponentially with time **t**.

$$\omega(t) = (Iy/Ix)*(\omega_1 + \omega_2) \approx \omega(0)*exp[-(0.0321/d)*t]$$

with the caliber **d** given here in **inches**.

The fast-mode and slow-mode exponential damping factors used herein are λ_1 and λ_2, respectively, in **inverse seconds**, such that, for example, the slow-mode coning angle amplitude α is given as a function of flight time **t** by:

$$\alpha(t) = \alpha(0)*exp[-\lambda_2*t] .$$

The damping with λ_1 of the fast-mode is modeled similarly, but because that nutation damping seems never to be a problem in rifle shooting, we shall ignore it for a time here. With regard to rifle bullets, the term *dynamic stability* really refers to the damping, λ_S or λ_2, of the slow-mode coning angle α.

Note that in accordance with modern engineering practice, we have reversed the signs of the damping factors λ_1 and λ_2 from those used in classic aeroballistics (λ_F and λ_S, respectively), and also that they are now specified per unit of time-of-flight **t** (in **seconds**) rather than per calibers **d** of path distance **s** travelled. Thus, when $\lambda_2 > 0$, the coning motion is damped, and for $\lambda_2 \leq 0$, the coning angle α is undamped,

and $\lambda_1 = -\lambda_F * V/d$ **(in inverse seconds)**
$\lambda_2 = -\lambda_S * V/d$ **(in inverse seconds)**

where **V = V(t) =** the instantaneous airspeed of the bullet, and
d = Caliber in **feet**.

If either damping factor, λ_1 or λ_2, were to go negative for some type of bullet and remained there for a significant time-of-flight **t**, the angular amplitude of either the fast-mode nutation or the coning angle α would increase without bound. When, for example, the amplitude of the coning angle α approaches **90 degrees**, we would certainly call that a bullet failure in dynamic stability.

Frictional Energy Loss of the Bullet

Yaw-drag *additionally* retards the forward motion of the bullet over its zero-yaw retardation due to its flying with the coning angle **α** as its long-term aerodynamic angle-of-attack. For a *dynamically stable* rifle bullet, some small fraction **e** of its kinetic energy of forward motion is continuously being bled off to reduce the amplitude **α** of that bullet's coning motion. In the absence of fast-mode nutation, the fraction **e** of that yaw-drag frictional energy loss goes toward frictional damping of the slow-mode coning motion. The frictional damping of the coning motion is ***not*** attributable to the orbital velocity of the CG of the bullet around its mean trajectory, but is due only to the forward motion of the coning bullet and to the coning angle as an aerodynamic angle-of-attack **α.**

Another small fraction **k** of the bullet's kinetic energy is also used to re-orient the axis of the coning motion in response to any step-change in the direction of approach of the apparent wind experienced by the flying bullet.

Since the CG of even a marginally stable bullet shows no measurable motion at the fast-mode nutation rate, there is almost no kinetic energy associated with that type of gyroscopic motion of the bullet's spin-axis direction or any associated *fast-mode coning motion* of the bullet's CG.

As kinetic energy is extracted from the orbital motion of the CG around its mean trajectory at the gyroscopic precession (or coning) rate **ω_2**, the orbital radius **r** of that orbit must decrease correspondingly to a lower orbital potential energy state, along with its associated coning angle **α**, with **$r = D*\mathrm{Sin}(\alpha)$**. This is the frictional damping mechanism which reduces the coning angle **α** over the distance **s** traversed by the bullet along the mean trajectory or, alternatively, over its time-of-flight **t** at airspeed **V(t)**.

Aerodynamic drag itself is a frictional force in that it can only act to oppose or retard bullet motion through the air. The total decrease in the bullet's kinetic energy due to this retardation of forward motion is caused only by the force of aerodynamic drag **F_D** experienced by that bullet at any time during its flight. The deceleration of the bullet is **F_D/m** at any point during its ballistic free flight. The ***direction*** of the aerodynamic drag force vector **F_D** is always "downwind" in the direction of movement of the apparent wind airstream approaching the bullet, and this direction is independent of the orientation of that bullet. Except for damping or re-orienting the coning motion and any nutation of the bullet, the remaining frictional energy loss is dissipated as heat energy. We shall explore this energy relationship below.

Additional Gyroscopic Stability Considerations

Various estimators for calculating the required rifling twist-rates for adequate gyroscopic stability **Sg** for rifle bullets have been used since the Greenhill Formula of the mid-1800's. We now also use Don Miller's formulation for VLD-type bullets and Bob McCoy's McGYRO calculations developed for artillery projectiles. A common feature of these estimators is that they all rely heavily upon the bullet length **L** in **calibers** as a basic slenderness ratio of the projectile. Some formulations also adjust for muzzle velocity, air density, and the average material density of the projectile.

Ideas are currently changing about what values of **Sg** constitute desired and adequate initial gyroscopic stability, especially in the flat-firing of rifles to extended ranges greater than 1000 yards—Extreme Long Range, or ELR shooting. Formerly, we considered an initial **Sg** of **1.2** to **1.4** to be adequate for best short-range rifle accuracy. We now realize the advantages of reducing aerodynamic drag in early flight by providing our conventional jacketed, lead-cored rifle bullets with an initial **Sg** of at least **1.5**, as recommended by Bob McCoy and Bryan Litz. Riflemen are currently learning to launch their new monolithic copper-alloy ultra-low-drag (ULD) bullets with an initial **Sg** of **2.5** to **3.5** for best results in ELR shooting. Achieving this initial **Sg** typically requires a rifling twist-rate of about **20 calibers per turn** for monolithic ULD bullets of about **5.5 calibers** in length **L**. The resulting higher bullet spin-rates $\boldsymbol{\omega_0}$ at high muzzle speeds $\mathbf{V_0}$ are not compatible with the use of conventional jacketed, lead-cored match rifle bullets. The jacketed bullets fail mechanically and disintegrate in mid-air.

New Analytical Calculations

Munk's Equation from early aerodynamics allows us to estimate the overturning moment **M** acting upon a slender solid-body-of-revolution in a laminar flow-field. Munk was a student of Prandtl in Germany in the early 20th Century. In our terms, Munk's Equation is

$$\mathbf{M = q*Sin(2*\alpha)*(Vol - S*Xcg)}$$

where **Vol = Volume of projectile**

Xcg = Distance from nose to CG of projectile.

The doubling of the aerodynamic angle-of-attack **α** comes directly from wind-tunnel observations of attached surface telltales (strips of yarn) for solids-of-revolution in laminar flow-fields. ULD rifle bullets fly with attached, laminar boundary layer flow-fields over their ogives at supersonic and subsonic airspeeds.

A paper published in 1989 by Ing. Dr. Beat P. Kneubuehl of Thun, Switzerland, entitled "What is the maximum length of a spin stabilized projectile?," details an analytical procedure for calculating the important aeroballistic parameters of **Iy/Ix** ratio, **CM_α**, and initial **Sg** from basic data for any reasonable projectile having a homogeneous mass distribution. I discovered this little gem of ballistics papers on Research Gate.

For a simple cone-on-cylinder projectile model, Kneubuehl analytically calculates the mass properties and, from Munk's Equation, the aeroballistic moment coefficient **CMα** of the projectile:

$$Wt(calc) = (\pi/4)*\rho_p*d^3 *L*(1 - 2*h/3) \quad \text{grains}$$

$$Ix = (\pi/32)*\rho_p*d^5 *L*(1 - 4*h/5) \quad \text{grain-inches}^2$$

$$Iy = (\pi/960)*\rho_p*d^5 *L*f_1(L,h) \quad \text{grain-inches}^2$$

$$Iy/Ix = f_1(L,h)/[30*(1 - 4*h/5)$$

$$CM\alpha = (\partial M/\partial \alpha)/(q*S*d) = L*f_2(h)$$

where **Wt(calc) = Calculated weight of projectile in grains**

d = Reference Diameter of bullet in inches = 1.0 caliber

ρ_p = Density of monolithic bullet in grains/cubic inch

L = Bullet length in calibers (including full-length nose)

L = Actual Bullet Length - LN + LFN

LFN = Full Length of the non-truncated ogival Nose

LN = Truncated Nose Length

h = LFN/L

$$f_1(L,h) = 15 – 12*h + L^2 *(60 – 160*h + 180*h^2 – 96*h^3 + 19*h^4)/(3 – 2*h)$$

and $$f_2(h) = (18 – 24*h + 7*h^2)/(18 – 12*h).$$

The *initial* gyroscopic stability **Sg** of this cone-on-cylinder projectile model can then also be analytically calculated as

$$Sg = 0.300*(\rho_p/\rho)*[Tan^2(180/n)]*(5 – 4*h)^2/[f_1(L,h)*f_2(h)]$$

where **ρ = Ambient air density**

n = Rifling twist-rate in calibers/turn.

The angular argument of the tangent function is just the helix angle of the rifling spiral within the barrel given by **180/n** in degrees or **π/n** in radians.

Adjusting the Swiss Formulation for ULD Rifle Bullets

The Swiss paper is primarily concerned with artillery shell ballistics. We have enough data from designing solid monolithic ULD rifle bullets to customize these analytical calculations for those and any similar rifle bullets of solid construction with homogeneous material density.

We can adjust $\mathbf{f_2(h)}$ by a factor of **2.62/3.41** (from data in the Swiss paper) to bring the analytically calculated overturning moment coefficient $\mathbf{CM_\alpha}$ into agreement with wind-tunnel data measurements for the cone-on-cylinder rifle bullet models, explicitly at **Mach 2.5** (and higher launch speeds), but somewhat applicable for slower supersonic airspeeds:

$$\mathbf{f_{2A}(h) = (2.62/3.41)*f_2(h)}$$

$$\mathbf{CM_\alpha = L*f_{2A}(h).}$$

These $\mathbf{CM_\alpha}$ adjustment values are for a 4.5-calibers long projectile model with **h = 0.57** (which is measured for our Mark II ULD bullets) at **Mach 2.5**. For a given rifle bullet, the aeroballistic overturning moment coefficient $\mathbf{CM_\alpha}$ tends to be approximately the same value for all high supersonic and subsonic airspeeds, but varies up and down slightly, especially around the transonic region.

The actual weight **Wt** of the a solid monolithic ULD rifle bullet in any caliber is **1.1418 times** the analytically calculated weight **Wt(calc)** for the corresponding cone-on-cylinder model of that same rifle bullet. The ULD bullet design utilizes either a secant ogive with **RT/R = 0.500** or a Sears-Haack lowest-drag nose shape and also features an aerodynamically effective boat-tail of about **0.7 calibers** in length. Since we know both of these weights, we can use their ratio **Wt/Wt(calc)** to adjust the analytically calculated ratio of second moments **Iy/Ix**:

$$\mathbf{\{Iy/Ix\}_A = 1.1418^{0.894} *f_1(L,h)/[30*(1 - 4*h/5)}$$

$$\mathbf{\{Iy/Ix\}_A = 1.12586*f_1(L,h)/[30*(1 - 4*h/5).}$$

This adjustment brings these analytical calculations of **Iy/Ix** into agreement with our numerical integrations of these mass properties for a wide array of different solid monolithic ULD rifle bullets.

We associate this weight-ratio adjustment with $\mathbf{f_1(L,h)}$, and adjust that analytical function accordingly:

$$f_{1A}(L,h) = 1.12586*f_1(L,h)$$

so that $$\{Iy/Ix\}_A = f_{1A}(L,h)/[30*(1 - 4*h/5)].$$

We can then separately adjust the individually calculated second moments of inertia, **Ix** and **Iy**, by the factor

$$1.1418^{1.037} = 1.147416$$

so that

$$\{Iy\}_A = 1.147416*(\pi/960)*\rho_p*d^5 *L*f_{1A}(L,h)$$

$$\{Ix\}_A = 1.147416*(\pi/32)*\rho_p*d^5 *L*(1 - 4*h/5)$$

with each second moment of inertia given in **grain-inches^2**.

Note that the ratio **Iy/Ix** remains as previously adjusted. These adjusted analytical calculations of the second moments of inertia, **Iy** and **Ix**, then agree closely with corresponding numerically integrated values for solid monolithic ULD rifle bullets of all calibers.

Finally, we can get good agreement with McGYRO initial **Sg** estimates and qualified agreement with Don Miller's VLD **Sg** estimates for solid monolithic ULD rifle bullets if we replace the initial constant factor of **0.300** in the **Sg** formulation with **0.2339**. This formulation is optimized for rifle bullets of 30- to 50-caliber. The adjusted analytic formulation of initial **Sg** then becomes

$$\{Sg\}_A=0.2339*(\rho_p/\rho)*[(5 - 4*h)*Tan(180/n)]^2/[f_{1A}(L,h)*f_{2A}(h)]$$

Both the air density **ρ** and projectile density $\boldsymbol{\rho_p}$ are used explicitly here, but no separate correction for muzzle velocity variations is available. This formulation calculates only the *initial* **Sg**, and does not work later in the flight.

We can invert this relationship to find the rifling twist-rate **n** in calibers per turn which will produce a desired initial gyroscopic stability **Sg** for this bullet.

$$Tan^2(\pi/n) = Sg*[f_{1A}(L,h)*f_{2A}(h)]/[0.2339*(\rho_p/\rho)*(5 - 4*h)^2]$$

$$Tan(\pi/n) = SQRT\{Sg*[f_{1A}(L,h)*f_{2A}(h)]/[0.2339*(\rho_p/\rho)]\}/(5 - 4*h)$$

$$\pi/n = ATAN\{SQRT\{Sg*[f_{1A}(L,h)*f_{2A}(h)]/[0.2339*(\rho_p/\rho)]\}/(5 - 4*h)\}$$

$$n = \pi/ATAN\{SQRT\{Sg*[f_{1A}(L,h)*f_{2A}(h)]/[0.2339*(\rho_p/\rho)]\}/(5 - 4*h)\}.$$

Dynamic Stability

In classic aeroballistics, the dynamic stability of a spin-stabilized projectile is given as

$$\mathbf{Sd = 2T/H = 2(PT)/(PH)}$$

$$\mathbf{Sd = 2(\omega_1\lambda_2 + \omega_2\lambda_1)/[(\omega_1 + \omega_2)(\lambda_1 + \lambda_2)]}$$

The conditions for simultaneous gyroscopic and dynamic stability are given as

$$\mathbf{Sg > 1/[Sd(2 - Sd)] > 1}$$

and $$\mathbf{0 < Sd < 2}$$

As **Sd** approaches **0**, the slow-mode (coning) motion becomes less damped, and as **Sd** approaches **2**, the fast-mode (nutation) motion becomes less damped.

Noting the symmetry about **Sd = 1**, we shall formulate a new expression for **(Sd – 1)** which is symmetric about **zero**. After some algebra, we find:

$$\mathbf{Sd - 1 = [(\omega_1 - \omega_2)/(\omega_1 + \omega_2)][(\lambda_2 - \lambda_1)/(\lambda_2 + \lambda_1)]}$$

$$\mathbf{Sd - 1 = [(R - 1)/(R + 1)][(\lambda_2 - \lambda_1)/(\lambda_2 + \lambda_1)]}$$

with $\mathbf{-1 < (Sd - 1) < +1}$ (for dynamic stability).

We know that $\boldsymbol{\omega_1 > \omega_2}$ because $\boldsymbol{\omega_1 = \omega_2}$ is the lower boundary condition on $\boldsymbol{\omega_1}$ for gyroscopic stability where **Sg = R = 1**.

Now suppose that the damping factors are such that the amplitude of each type of angular motion is reduced by the *same damping fraction* during each cycle of its motion. In this case, we would have

$$\boldsymbol{\lambda_1/\omega_1 = \lambda_2/\omega_2}$$

or $$\boldsymbol{\lambda_1 = (\omega_1/\omega_2)*\lambda_2 = R*\lambda_2}.$$

Then, after more algebra

$$\mathbf{Sd - 1 = -[(R - 1)/(R + 1)]^2}$$

or $$\mathbf{Sd = 1 - [(R - 1)/(R + 1)]^2}.$$

In this special case, the formulation shows that

$$\mathbf{0 < Sd < 1}$$

whenever the gyroscopic stability ratio **R > 1**, and the fast-mode motion (nutation) is certainly always well damped.

The rifle bullet motions observed in 6 degree-of-freedom flight simulations often tend to behave in just about this way, and there might even be sound physical reasons behind this type of behavior. With initial **R**-values of about **5** to **10** for rifle bullets which are gyroscopically well stabilized with initial **Sg** from **1.8** to **3.0**, the fast-mode cycles occur just **R**-times more frequently in time than do the coning cycles. With the $\boldsymbol{\lambda_1}$ fast-mode damping factor also **R**-times larger than $\boldsymbol{\lambda_2}$, the nutations would damp to insignificance $\mathbf{R^2}$-times more quickly over flight time than does the coning motion in this special case. No lack of adequate fast-mode damping $\boldsymbol{\lambda_1}$ seems ever to be observed for rifle bullets fired with adequate initial gyroscopic stability **Sg**.

The table below shows this special case relationship for values of **R** from **1** to **37**, where **Sg = 9.757.** Note that the conditions for both gyroscopic and dynamic stability are continually met in this special case. As **R** continually increases during the flight, the dynamic stability **Sd** decreases toward **0**, or slow-mode instability. This indicates that the amplitude of the slow-mode coning motion **α(t)**, while becoming less damped as **R** increases, is never completely undamped in this special case.

For Constant Damping Ratio per Cycle: **λ1/λ2 = ω1/ω2**

Sd = 1 - ((R - 1)/(R + 1))^2

Sg	R	Sd	1/(Sd*(2-Sd))	1/Sg<	Sd*(2-Sd)
1.000	1.000	1.0000	1.0000	1.0000	1.0000
1.125	2.000	0.8889	1.0125	0.8889	0.9877
1.333	3.000	0.7500	1.0667	0.7500	0.9375
1.563	4.000	0.6400	1.1489	0.6400	0.8704
1.800	5.000	0.5556	1.2462	0.5556	0.8025
2.042	6.000	0.4898	1.3519	0.4898	0.7397
2.286	7.000	0.4375	1.4629	0.4375	0.6836
2.531	8.000	0.3951	1.5772	0.3951	0.6340
2.778	9.000	0.3600	1.6938	0.3600	0.5904

3.025	10.000	0.3306	1.8120	0.3306	0.5519
3.273	11.000	0.3056	1.9314	0.3056	0.5177
3.521	12.000	0.2840	2.0518	0.2840	0.4874
3.769	13.000	0.2653	2.1729	0.2653	0.4602
4.018	14.000	0.2489	2.2945	0.2489	0.4358
4.267	15.000	0.2344	2.4165	0.2344	0.4138
4.516	16.000	0.2215	2.5389	0.2215	0.3939
4.765	17.000	0.2099	2.6617	0.2099	0.3757
5.014	18.000	0.1994	2.7846	0.1994	0.3591
5.263	19.000	0.1900	2.9078	0.1900	0.3439
5.513	20.000	0.1814	3.0312	0.1814	0.3299
5.762	21.000	0.1736	3.1547	0.1736	0.3170
6.011	22.000	0.1664	3.2784	0.1664	0.3050
6.261	23.000	0.1597	3.4021	0.1597	0.2939
6.510	24.000	0.1536	3.5260	0.1536	0.2836
6.760	25.000	0.1479	3.6500	0.1479	0.2740
7.010	26.000	0.1427	3.7740	0.1427	0.2650
7.259	27.000	0.1378	3.8981	0.1378	0.2565
7.509	28.000	0.1332	4.0223	0.1332	0.2486
7.759	29.000	0.1289	4.1465	0.1289	0.2412
8.008	30.000	0.1249	4.2708	0.1249	0.2341
8.258	31.000	0.1211	4.3951	0.1211	0.2275
8.508	32.000	0.1175	4.5195	0.1175	0.2213
8.758	33.000	0.1142	4.6439	0.1142	0.2153
9.007	34.000	0.1110	4.7684	0.1110	0.2097
9.257	35.000	0.1080	4.8928	0.1080	0.2044
9.507	36.000	0.1052	5.0174	0.1052	0.1993
9.757	37.000	0.1025	5.1419	0.1025	0.1945

Achieving Minimum Coning Angle Flight

David Tubb launched our new 225-grain copper 338-caliber ULD bullets in his 1000-yard testing with an initial **R**-value of **8.86** (**Sg = 2.75**) from his 7.5-inch twist Schneider 338 barrel. The values of **R** and **Sg** could only have increased during their supersonic flight to his 1000 yard target. The resulting rather surprisingly high ballistic coefficient **BC(G1)** measurement of **0.794** for an average airspeed of **Mach 2.46** indicates that these bullets were flying with a "minimum coning angle" motion (**$\alpha \approx \Delta\Phi << 0.1$ degree**) during all (or most) of their 1000-yard flights and certainly that they were dynamically stable (**$\lambda_2 > 0$**) all the way from launch to that target distance.

Bob McCoy's McDRAG program estimated a **BC(G1)** value of **0.703** for this bullet at **Mach 2.5**. David was able to measure the aerodynamic drag of these test bullets due solely to their zero-yaw coefficient of drag **CD_0**. I propose that we term this flight mode "***hyper-stable flight***." Otherwise, the bullet drag measurements from the usual firing tests (as calculated in McDRAG) apparently contain significant yaw-drag contributions. The bullet's actual coning angle-of-attack **α** is typically difficult and expensive to measure in flight.

For our 338-caliber prototype 225-grain ULD bullets launched with an initial **Sg = 2.75**, David Tubb measured an average **BC(G1)** which was **12.4 percent** higher than calculated by McDRAG at an average airspeed of **Mach 2.46**. David repeated these measurements for a 242-grain version of this same bullet, launched with an initial **Sg = 2.44**, and he measured an average **BC(G1)** which was **11.2 percent** higher than predicted by McDRAG at **Mach 2.44** average airspeed. The 225-grain version had 17 grains of copper removed by base-drilling in order to increase their initial **Sg** values when fired from rifles made with slower-twist barrels. This data indicates that the bullets fired with initial **Sg = 2.75** were significantly more "hyper-stable" than those fired at only **Sg = 2.44**. More test measurements should indicate the true threshold in initial gyroscopic stability **Sg** to achieve truly hyper-stable flight right out of the muzzle with these copper ULD bullets.

In ***minimum coning angle (hyper-stable) flight***, the increase **$\Delta\Phi$** in coning angle **α** due to the gravitational downward curving of the mean trajectory during each half coning cycle is closely matched by the exponential slow-mode damping $\boldsymbol{\lambda_2}$ of the coning angle $\boldsymbol{\alpha}$ during that same half coning cycle:

$$\alpha \leq (\alpha + \Delta\Phi)*exp[-\lambda_2*\pi/\omega_2]$$

This condition quickly produces a steady-state coning angle of **$\alpha \geq \Delta\Phi <<$ 0.1 degree** whenever $\boldsymbol{\lambda_2 \geq [\lambda_2]_{Min}}$ where**:**

$$[\lambda_2]_{Min} = (\omega_2/\pi)*ln[(\alpha + \Delta\Phi)/\alpha]$$

or $$[\lambda_2]_{Min} = 2*f_2*ln[(\alpha + \Delta\Phi)/\alpha].$$

As the damped coning angle $\boldsymbol{\alpha}$ approaches the change in flight path angle $\boldsymbol{\Delta\Phi}$ during this half coning cycle as its *lower limit*,

$$ln[(\alpha + \Delta\Phi)/\alpha] \leq ln[2] = 0.693147.$$

Then, for the 225-grain bullet test-fired, for which the initial coning frequency $\boldsymbol{f_2}$ and the initial coning period $\boldsymbol{T_2}$ are given by

$$f_2 = [(12*3378 \text{ fps}/7.5 \text{ in.})/12.1227]/9.86 = 45.22 \text{ hz}$$

$$T_2 = 1/f_2 = 1/45.22 = 22.114 \text{ milliseconds}$$

where $$Iy/Ix = 12.1227$$

$$R + 1 = 9.86.$$

Then $$[\lambda_2]_{Min} \leq 2*f_2*\ln(2) = 62.7 \text{ seconds}^{-1}$$

If $\boldsymbol{\lambda_2 = [\lambda_2]_{Min}}$, the time constant for damping reduction of the size of the coning angle $\boldsymbol{\alpha}$ by a factor of **1/e = 0.3679** is **15.95 milliseconds**, or **0.7213 coning cycles** (or **53.9 feet** of early flight).

An under-damped driven linear system continues oscillating at its driving frequency, but at a reduced amplitude. An over-damped driven linear system stops oscillating at that frequency, but never quite achieves equilibrium either. A critically damped linear system stops its oscillation as quickly as possible.

Here, we are dealing with a harmonic 2-dimensional rotational motion of the CG of the coning bullet. Only the radial angular magnitude $\boldsymbol{\alpha}$ of this coning motion is being frictionally damped with ongoing time-of-flight or flight distance. The rotational velocity of the coning bullet is ***not*** the cause of the frictional damping of the coning angle **α**. The rotational rate of the coning motion **ω_2** is completely independent of its amplitude **α**.

Aside:

But is this damping of the coning angle **α** really independent of the orbital velocity **v** of the CG of the bullet?

$$v = r*\omega_2 = D*Sin(\alpha)*\omega_2 \approx \alpha*D*\omega_2$$

The coning bullet is moving laterally (sideways) through the air at a subsonic airspeed **v** and presenting its largest possible cross-sectional area to that airflow. The frontal cross-sectional area is

$$S = (\pi/4)*d^2$$

If a rifle bullet has a volume **Vol** of about **3.15 cubic calibers**, as with many monolithic VLD and ULD bullets, its minimum side aspect area **Sa** would be **La*d**, where **La** is given by

$$La = Vol/S = 3.15*d^3/[(\pi/4)*d^2] = (12.6/\pi)*d = 4.011*d$$

and
$$Sa = La*d = 4.011*d^2 = 5.107*S.$$

If the subsonic coefficient of drag **CD** for this rifle bullet is about **0.100**, the drag force **F_{DC}** due to this coning motion would be

$$F_{DC} = (\rho/2)*[v^2\ ]*Sa*CD = (\rho/2)*[\alpha^2 *(D*\omega_2)^2\ ]*0.511*S$$

The orbital kinetic energy loss per half coning cycle **ΔE_C** would be

$$\Delta E_C = F_{DC}*\pi*r = \pi*(\rho/2)*\alpha^3 *D^3 *(\omega_2)^2 *0.511*S$$

But, as will be shown later, the loss in orbital potential energy with **α** per half coning cycle is

$$\Delta E_C = [m*(D*\omega_2)^2]*\alpha*\Delta\alpha.$$

Setting these energy losses equal, we have

$$\Delta\alpha = \alpha^2 *(\pi/m)*(\rho/2)*D*0.511*S$$

$$\Delta\alpha = \alpha*[1 - exp(-\lambda_C*T_2/2)]$$

$$exp(-\lambda_C*T_2/2) = 1 - \alpha*[(\pi/m)*(\rho/2)*D*0.511*S]$$

$$\lambda_C = (-2*f_2)*ln\{1 - \alpha*[0.802*(\rho/m)*D*S]\}.$$

$$\lambda_C = (-2*f_2)*\ln\{1 - \alpha*[0.630*(\rho/m)*d^2 *D]\}.$$

For a **250-grain 338 bullet** in an ICAO sea-level atmosphere (**ρ = 0.0764742 lbm/ft^3),**with **$f_2 \approx$ 45 hertz**, **D ≈ 4*d**, and **$\alpha \approx$ 0.100 radians**,

$$\lambda_C = (-2*f_2)*\ln\{1 - \alpha*[0.630*(\rho/m)*D*d^2]\}.$$

$$\lambda_C = -90\ hz*\ln[1 – 0.100*[0.630*2.1413*4*(0.028167)^3]$$

$$\lambda_C = -90\ hz*\ln[1 – 0.000012058]$$

$$\lambda_C = -90\ hz*[-0.000012058] = 0.0010852\ seconds^{-1} .$$

Critical damping would have a time constant of **921.5 seconds** (or **15.36 minutes** of flight time) at this tiny **λ_C** damping rate.

Since we are considering flight times of only a few seconds, we should be justified in saying that the damping factor **λ_2** affects only the angular size **α** of the coning motion and ***not*** the orbital motion of the CG. **[End of aside.]**

Minimum coning angle flight is achieved earlier in the bullet's flight when the bullet is perfectly launched with **zero** initial yaw and yaw-rate, when the initial spin-rate of the bullet is very high (**≈ 6000 revolutions/second**), when the bullet design is easier to stabilize gyroscopically (initial **Sg ≈ 3.0**), when crosswinds are light and steady, when the density of the ambient atmosphere is relatively low, and when bullet's launch velocity is very high.

Importantly, the coning motion of the bullet during this "minimum coning angle" hyper-stable flight mode still allows the rotational cancellation of the aerodynamic lift force acting on the bullet due to any crosswinds. Windage corrections would have to be at least an order of magnitude greater if this were not the case. Windage corrections remain attributable only to the aerodynamic drag force as first formulated by DeDion in 1859.

Ordinary outdoor test-firing for most bullets allows measurement of a total aerodynamic drag force which includes a significant yaw-drag component due to coning angles-of-attack often in the **2** to **10-degree** range. This coning angle is effectively a long-term aerodynamic angle-of-attack, but these small attitude angles are difficult and expensive to measure in flight, especially in outdoor firing tests. We understand that firing-test measured aerodynamic drag is reduced somewhat merely by increasing the fired bullet's initial gyroscopic stability from a marginal **Sg** of **1.4** to a nominal **Sg** of **1.5**. It stands to reason that increasing the initial **Sg** a bit more might decrease measured aerodynamic drag even more.

Hyper-stabilizing our test-fired bullets with an initial **Sg** of **2.75** in David Tubb's test-firings effectively allowed their pure **CD_0** aerodynamic drag coefficient to be measured for the average Mach-speed over the entire flight to the target (**Mach 2.46**). However, increasing initial **Sg** even further should *not* be expected to provide very much (if any) additional reduction in measured aerodynamic drag below the **CD_0** coefficient for zero-yaw flight.

Energy Considerations

The total energy **TE** of the fired rifle bullet which is conserved in ballistic flight is

$$\mathbf{TE = E + E_C + P_C + mgh + Heat}$$

where $\mathbf{E =}$ Kinetic energy of the bullet due to forward motion

$\mathbf{E_C =}$ Kinetic energy of the coning motion

$\mathbf{P_C =}$ Potential energy of the coning motion

mgh = Gravitational potential energy of bullet

Heat = Dissipated energy absorbed by surroundings.

Here, we are only interested in the first three energy terms on the right-side, since the height **h** is essentially constant in flat-firing and the eventual heating of the environment cannot be measured.

We can gain additional insight into this steady-state "minimum coning angle" hyper-stable flight by looking at the bullet's loss of kinetic energy **E** in flight. Let us say that at any time **t** during flight, the loss in kinetic energy **ΔE** over the small time interval **Δt** is governed by the Equations of Motion as:

$$\mathbf{E(t + \Delta t) = E(t) - \Delta E(\Delta t)}$$

and

$$\mathbf{\Delta E(\Delta t) = F_D{*}\Delta s = F_D{*}V{*}\Delta t}$$

where $\mathbf{F_D}$ is the total aerodynamic force of drag and **Δs** is the path length (in feet) travelled during the small time-interval **Δt** seconds.

In particular, we are interested in the loss in kinetic energy **ΔE** during any particular half coning cycle where

$$\mathbf{\Delta t = (2\pi/\omega_2)/2 = 1/(2{*}f_2) = T_2/2 \text{ seconds}}.$$

One half of the period $\mathbf{T_2}$ of the coning motion is the time interval during which the coning motion adjusts to any step-change in the direction of the apparent wind approaching the flying bullet.

In linear aeroballistics theory, $\mathbf{F_D}$ is accurately modelled as

$$\mathbf{F_D = q{*}S{*}(CD_0 + \delta^2 {*}CD_\alpha + \delta^4 {*}CD_4 + ...)}$$

where $\boldsymbol{\delta = \mathrm{Sin}(\alpha) \approx \alpha}$ **(in radians).**

As aerodynamic drag must remain an *even function* in $\boldsymbol{\delta}$, only even powers of $\boldsymbol{\delta}$ can appear in this series expansion for $\mathbf{F_D}$. Here, we use only the first two terms of this series.

$\mathbf{CD_0}$ is the coefficient of minimum drag for exactly nose-forward aerodynamic flight at a given airspeed (Mach Number), and $\mathbf{CD_\alpha}$ is the $\boldsymbol{\delta^2}$ yaw-drag coefficient at that same airspeed.

Now the expression for kinetic energy loss in any particular half coning cycle becomes

$$\Delta E(T_2/2) = q{*}S{*}(CD_0 + \alpha^2 {*}CD_\alpha){*}V{*}T_2/2$$

The sensitivity of this expression to coning angle $\boldsymbol{\alpha}$ is given by its partial derivative with respect to $\boldsymbol{\alpha}$:

$$\partial(\Delta E)/\partial\alpha = 2{*}\alpha{*}q{*}S{*}V{*}CD_\alpha{*}T_2/2$$

$$\partial(\Delta E)/\partial\alpha = \alpha{*}q{*}S{*}V{*}CD_\alpha/f_2.$$

where $\mathbf{f_2 = 1/T_2 = \omega_2/2\pi}$ **= Coning rate in hertz.**

Coning Kinetic Energy

We can also formulate the much smaller kinetic energy E_C of the orbital coning motion itself as

$$E_C = (m/2)*(r*\omega_2)^2 = (m/2)*(D*Sin\alpha*\omega_2)^2$$

$$E_C = (m/2)*(D*\omega_2)^2 *\alpha^2$$

where **D** is the slowly varying coning distance of the CG of the bullet from its coning apex. Here we are again using the small angle approximation for small coning angles **Sinα ≈ α (in radians)**.

Forming the partial derivative again with respect to **α**, we find the sensitivity of E_C to **α** to be

$$\partial(E_C)/\partial\alpha = m*(D*\omega_2)^2 *\alpha.$$

We now reason that the kinetic energy ΔE_C of the coning motion lost to frictional damping of the coning angle by the difference **Δα** during each of these "steady-state" half coning cycles must be a small fraction **e** of the kinetic energy **ΔE** extracted from the forward motion of the bullet due to that same coning angle difference **Δα** during that same half coning cycle.

For **(Δα, α) ≠ 0**, we can write

$$e*\Delta\alpha*\partial(\Delta E)/\partial\alpha = \Delta\alpha*\partial(E_C)/\partial\alpha$$

and

$$e*(\alpha*2\pi*q*S*V/\omega_2)*CD_\alpha = \alpha*m*(D*\omega_2)^2$$

$$e*(2\pi*q*S*V/\omega_2)*CD_\alpha = m*(D*\omega_2)^2$$

From Coning Theory, we know that the distance **D** (in feet) is given by

$$D = q*S*(CL_\alpha + CD_0)/[m*(\omega_2)^2]$$

or

$$D*\omega_2 = q*S*(CL_\alpha + CD_0)/[m*\omega_2].$$

Substituting for $D*\omega_2$ and simplifying, we have

$$e*(2\pi*q*S*V/\omega_2)*CD_\alpha = m*(q*S)^2*(CL_\alpha + CD_0)^2/[m*\omega_2]^2$$

or

$$e*(2\pi*m*V/\omega_2)*CD_\alpha = (q*S)*(CL_\alpha + CD_0)^2/[\omega_2]^2$$

and

$$e*(2\pi*m*V*\omega_2)*CD_\alpha = (q*S)*(CL_\alpha + CD_0)^2$$

Also, from Coning Theory, we know that the magnitude of the coning rate $\boldsymbol{\omega_2}$ is given by

$$\omega_2 = q*S*d*CM_\alpha/(I_x*\omega)$$

where $I_x = m*d^2*k_x^2$

and $\mathbf{k_x}$ **=** Radius of Gyration of the bullet's mass distribution about its **x**-axis given in units of calibers (**d, in feet**).

Substituting for $\boldsymbol{\omega_2}$ and simplifying, we have

$$(q*S*d*CM_\alpha)*(e*2\pi*m*V*CD_\alpha) = q*S*(CL_\alpha + CD_0)^2 *(\omega*m*d^2*k_x^2)$$

$$e*2\pi*V*CM_\alpha*CD_\alpha = (\omega*d*k_x^2)*(CL_\alpha + CD_0)^2$$

or

$$(\omega*d*k_x^2)/(e*2\pi*V) = CM_\alpha*CD_\alpha/(CL_\alpha + CD_0)^2.$$

But, the auxiliary parameter **P** given in radians of bullet rotation per caliber of bullet travel in classic aeroballistics is given by **Eq. 50** as

$$P = (I_x/I_y)*p*d/V = (k_x/k_y)^2 *\omega*d/V$$

So,

$$(\omega*d*k_x^2)/(e*2\pi*V) = [k_y^2/(e*2\pi]*P$$

And,

$$[k_y^2/(e*2\pi]*P = CM_\alpha*CD_\alpha/(CL_\alpha + CD_0)^2$$

Or,

$$P = e*2\pi*k_y^{-2} *CM_\alpha*CD_\alpha/(CL_\alpha + CD_0)^2.$$

And,

$$\omega*d/V = e*2\pi*k_x^{-2} *[k_y^2/(e*2\pi]*P = (I_y/I_x)*P$$

$$\omega*d/V = e*2\pi*k_x^{-2} *CM_\alpha*CD_\alpha/(CL_\alpha + CD_0)^2$$

$$\omega = e*2\pi*k_x^{-2} *(V/d)*CM_\alpha*CD_\alpha/(CL_\alpha + CD_0)^2$$

In particular, right out of the muzzle at **t = 0**,

$$\omega_0 = 2\pi^*V_0/Tw = 2\pi^*V_0/(n^*d)$$

So,

$$\omega_0{}^*d/V_0 = 2\pi/n = (I_y/I_x)^*P_0$$

And,

$$n = 2\pi/[(I_y/I_x)^*P_0] = 2\pi^*(k_y/k_x)^{-2} /[e^*2\pi^*k_y{}^{-2} {}^*CM_\alpha{}^*CD_\alpha/(CL_\alpha + CD_0)^2$$

$$n = (k_x{}^2/e)^*(CL_\alpha + CD_0)^2/(CM_\alpha{}^*CD_\alpha)$$

or

$$e = (k_x{}^2/n)^*(CL_\alpha + CD_0)^2/(CM_\alpha{}^*CD_\alpha).$$

For the well studied 30-caliber 168-grain Sierra International bullet, for example, at an initial airspeed of Mach 2.5:

$$\begin{aligned} L &= 3.98 \text{ calibers} \\ k_x{}^{-2} &= 9.218 \text{ calibers}^{-2} \\ CL_\alpha &= 2.850 \\ CD_0 &= 0.320 \\ CM_\alpha &= 2.560 \\ CD_\alpha &= 4.400 \\ CM_\alpha{}^*CD_\alpha/(CL_\alpha + CD_0)^2 = CS_\alpha &= 1.121 \end{aligned}$$

For brevity we are coining the Coefficient of Stability CS_α for this expression combining the four conventional aeroballistic coefficients.

Greenhill's formula suggests a barrel twist-rate **n** for this bullet of either

$$n = 150/3.98 = 37.7 \text{ calibers (up to Mach 2.5).}$$

Substituting these values into our expression for **e** above:

$$e = (k_x{}^2/n)^*/CS_\alpha = 0.002567$$

and

$$1/e = 389.5.$$

for achieving initial coning motion having “minimum coning angle” with this bullet right out of the muzzle of the rifle barrel at Mach 2.5.

Then, since $\mathbf{k_x^{-2} = 9.0\ calibers^{-2}}$ for almost any modern monolithic rifle bullet, the maximum value of **n** for achieving initial hyper-stable flight is:

$$\mathbf{n = (389.54/k_x^{-2})/CS_\alpha}$$
$$\mathbf{n = 43.28/CS_\alpha.}$$

Coning Potential Energy

The orbital potential energy $\mathbf{P_C}$ of the coning rifle bullet with $\mathbf{F_C = -k_C{*}r}$ and $\mathbf{r = D{*}Sin(\alpha)}$ can be formulated, with $\mathbf{Sin(\alpha) \approx \alpha}$, as

$$\mathbf{P_C = -\int F_C{*}dr = k_C{*}r^2/2 = q{*}S{*}Sin(\alpha){*}[CL\alpha+CD]{*}r/2}$$

or $$\mathbf{P_C = q{*}S{*}D{*}[CL\alpha+CD]{*}\alpha^2/2.}$$

Because the harmonic coning motion is isotropic and the orbital motion of the CG of the bullet is circular (at least non-elliptical), the orbital kinetic energy $\mathbf{E_C}$ and orbital potential energy $\mathbf{P_C}$ are ***always equal***:

$$\mathbf{E_C = (m/2){*}(D{*}\omega_2)^2\ {*}\alpha^2 =}$$

$$\mathbf{P_C = q{*}S{*}D{*}[CL\alpha+CD]{*}\alpha^2/2}$$

or $$\mathbf{m{*}D{*}(\omega_2)^2 = q{*}S{*}[CL\alpha+CD].}$$

$$\mathbf{D = q{*}S{*}(CL_\alpha + CD_0)/[m{*}(\omega_2)^2].}$$

Recall from *Coning Theory* that

$$\mathbf{D = q{*}S{*}(CL_\alpha + CD_0)/[m{*}(\omega_2)^2] \qquad QED.}$$

Since this coning distance parameter **D** is so basic to Coning Theory, perhaps we should simplify its aeroballistic definition here.

From Coning Theory, we know that

$$\mathbf{\omega_2 = q{*}S{*}d{*}CM_\alpha/(Ix{*}\omega)}$$

and from Tri-Cyclic Theory, we know that

$$\mathbf{(Ix/Iy){*}\omega = \omega_2 + \omega_1 = \omega_2{*}(R + 1)}$$

So, $$\mathbf{L = Ix{*}\omega = Iy{*}\omega_2{*}(R + 1).}$$

Then, $$\mathbf{\omega_2 = q{*}S{*}d{*}CM_\alpha/[Iy{*}\omega_2{*}(R + 1)]}$$

And, $$\mathbf{\omega_2^2 = q{*}S{*}d{*}CM_\alpha/[Iy{*}(R + 1)].}$$

Substituting into our expression for **D** above, and simplifying, we have

$$\mathbf{D = q*S*(CL_\alpha + CD_0)/[m*(\omega_2)^2].}$$

$$\mathbf{D = [Iy*(R + 1)]*[q*S*(CL_\alpha + CD_0)]/[m*q*S*d*CM_\alpha]}$$

$$\mathbf{D = [(m*d^2 *ky^2)*(R + 1)/(m*d)]*[(CL_\alpha + CD_0)/CM_\alpha]}$$

$$\mathbf{D = [(d*ky^2)*(R + 1)]*[(CL_\alpha + CD_0)/CM_\alpha]}$$

$$\mathbf{D = [d*kx^2 *(ky^2/kx^2)*(R + 1)]*[(CL_\alpha + CD_0)/CM_\alpha]}$$

$$\mathbf{D = [(d*kx^2)*(Iy/Ix)*(R + 1)]*[(CL_\alpha + CD_0)/CM_\alpha].}$$

Finally, the coning distance **D** expressed in calibers **d** can be written as

$$\mathbf{D/d = [(Iy/Ix)/kx^{-2}]*[(R + 1)*(CL_\alpha + CD_0)/CM_\alpha].}$$

The first bracketed expression is fixed for each type of rifle bullet. The ratio of its second moments of inertia (**Iy/Ix**) is about **7** to **15**, with about **7** to **10.5** being typical for jacketed lead-core match bullets. and **12** to **15** being typical for longer CNC-turned monolithic ULD bullets. The inverse of the square of the radius of gyration about the spin-axis in calibers ($\mathbf{kx^{-2}}$) is always about **9.2** $\mathbf{calibers^{-2}}$ for jacketed, tangent-ogive rifle bullets and about **9.0** $\mathbf{calibers^{-2}}$ for monolithic secant-ogive ULD bullets.

The Stability Ratio **R** and the three aeroballistic coefficients in the second set of brackets are to be evaluated at any time during the flight as **D/d** gradually lengthens during the bullet’s flight downrange.

Evaluation of Slow-Mode Damping Factor

As shown above, the kinetic energy loss due to yaw-drag ΔE_α over a half coning cycle $T_2/2$ can be written as

$$\Delta E_\alpha(T_2/2) = q*S*V*\alpha^2 *CD_\alpha*T_2/2.$$

We can also formulate the kinetic energy E_C of the orbital coning motion itself as

$$E_C = (m/2)*(r*\omega_2)^2 = (m/2)*(D*Sin\alpha*\omega_2)^2$$

$$= (m/2)*(D*\omega_2)^2 *\alpha^2$$

where **r** is the coning radius of the CG of the bullet orbiting around a "mean CG" location moving smoothly along the "mean trajectory" of the bullet at its "mean velocity," and **D** is the slowly varying coning distance of the CG of the bullet from its coning apex, each given in feet, so that $r = D*Sin(\alpha)$.

Now, as the coning angle α decreases (due to frictional damping) from its initial value α_0 to its final value α_1 at the completion of this half coning cycle, the change ΔE_C in orbital coning energy can be written as

$$\Delta E_C = (m/2)*(D*\omega_2)^2 *(\alpha_0^2 - \alpha_1^2)$$

$$\Delta E_C = m*(D*\omega_2)^2 *[(\alpha_0 + \alpha_1)/2]*(\alpha_0 - \alpha_1)$$

$$\Delta E_C = m*(D*\omega_2)^2 *\alpha*\Delta\alpha$$

where $(\alpha_0 + \alpha_1)/2 = \alpha$, the *average* coning angle over this half cycle,

and $\alpha_0 - \alpha_1 = \Delta\alpha > 0$, the *reduction* in coning angle due to damping.

We now hypothesize that, at least in **hyper-stable flight** in which no nutation needs damping, and for **dynamically stable bullets**, the average loss in "forward motion" kinetic energy ΔE_α over any half coning cycle due to flying with an aerodynamic angle-of-attack α *causes* the average "frictional damping" decrease in coning energy ΔE_C during that same half coning cycle. Therefore, these two energy losses must be proportional to each other. That is to say, we are tentatively assuming that a small fraction **e** of the yaw-drag of the bullet directly causes the damping of its coning angle α in *steady-state, minimum coning angle, hyper-stable flight*.

If this hypothesis is true, we can set $\Delta E_C = e*\Delta E_\alpha$ over any particular half coning cycle, where the constant fraction **e** is greater than **zero** but not greater than **1.0**, and so that

$$m*(D*\omega_2)^2 *\alpha*\Delta\alpha = q*S*V *\alpha^2 *e*CD_\alpha*T_2/2$$

or, dividing through by α^2 and by $[m*(D*\omega_2)^2]$,

$$(\Delta\alpha)/\alpha = (T_2/2)*[q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2].$$

We recognize this expression as having the form of the classic exponential damping of the coning angle α which was discussed above:

$$\alpha(t) = \alpha(0)*\exp[-\lambda_2*t]$$

with $$\lambda_2 = [q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2].$$

If we replace the half coning period $T_2/2$ with a small increment in time **dt**, and replace $\Delta\alpha$ per half coning cycle with a small decrement $-d\alpha$ in α, then in the limit as **dt** approaches zero, this expression becomes

$$d\alpha/\alpha = -\lambda_2*dt$$

After integrating both sides from **0** to **t**,

$$\ln[\alpha(t)/\alpha(0)] = -\lambda_2*t$$

Or, after exponentiating

$$\alpha(t) = \alpha(0)*\exp[-\lambda_2*t] \qquad [QED].$$

Thus, we have derived the long-accepted damping relationship from the basic physics of our hypothesis that a portion **e** of the yaw-drag causes the damping of the coning angle for dynamically stable bullets in hyper-stable flight.

If only a small fraction **e** $(0 < e \leq 1.0)$ of this extra yaw-drag induced kinetic energy loss is actually responsible for frictional damping of the coning angle $\alpha(t)$, we accommodate that simply by using $e*CD_\alpha$ in the above expression for λ_2:

$$\lambda_2 = [q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2].$$

Two basic magnitude relations from Coning Theory allow simplification of this expression above for the slow-mode damping factor $\boldsymbol{\lambda_2}$. We know that the coning distance **D** (in feet) is given by

$$D = q*S*(CL_\alpha + CD_0)/[m*(\omega_2)^2]$$

and, we know that the magnitude of the coning rate $\boldsymbol{\omega_2}$ is given by

$$\omega_2 = q*S*d*CM_\alpha/(I_x*\omega)$$

where, from Tri-Cyclic Theory, the expression for angular momentum of the spinning bullet can be written as

$$I_x*\omega = I_y*(\omega_2 + \omega_1) = I_y*\omega_2*(R + 1).$$

Substituting in the denominator of the expression for $\boldsymbol{\omega_2}$:

$$\omega_2 = q*S*d*CM_\alpha/[I_y*\omega_2*(R + 1)]$$

or, multiplying by $\boldsymbol{\omega_2}$

$$(\omega_2)^2 = q*S*d*CM_\alpha/[I_y*(R + 1)].$$

Now, we can reformulate the coning distance **D** as

$$D = I_y*(R + 1)*q*S*(CL_\alpha + CD_0)/[m*q*S*d*CM_\alpha]$$

$$D = I_y*(R + 1)*(CL_\alpha + CD_0)/(m*d*CM_\alpha).$$

And, $(\boldsymbol{D*\omega_2})^2$ can be expressed as

$$(D*\omega_2)^2 = \{[I_y*(R + 1)*(CL_\alpha + CD_0)]^2 *q*S*d*CM_\alpha\}/\{(m*d*CM_\alpha)^2 *I_y*(R + 1)\}$$

$$(D*\omega_2)^2 = \{q*S*I_y*(R + 1)*(CL_\alpha + CD_0)^2\}/\{m^2 *d*CM_\alpha\}$$

Substituting for $(\boldsymbol{D*\omega_2})^2$ in our expression for $\boldsymbol{\lambda_2}$, we have

$$\lambda_2 = [q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2]$$

$$\boldsymbol{\lambda_2 = \{(m^2 * d * CM_\alpha) * [q * S * V * e * CD_\alpha]\} / \{m * q * S * I_y * (R + 1) * (CL_\alpha + CD_0)^2\}}$$

Collecting terms

$$\boldsymbol{\lambda_2 = \{m * d * e * V / [I_y * (R + 1)]\} * \{CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2\}}$$

Let a Stability Coefficient $\boldsymbol{CS_\alpha}$ stand for the combined aeroballistic coefficients expression for any particular Mach speed:

$$\boldsymbol{CS_\alpha = CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2}$$

Then

$$\boldsymbol{\lambda_2 = \{m * d * e * V / [I_y * (R + 1)]\} * CS_\alpha}$$

From Tri-Cyclic Theory

$$\boldsymbol{(I_x / I_y) * \omega = \omega_1 + \omega_2 = \omega_2 * (R + 1)}$$

so, $\boldsymbol{R + 1 = (I_x / I_y) * \omega / \omega_2 = (I_x / I_y) * f / f_2}$

and $\boldsymbol{I_y * (R + 1) = I_x * f / f_2}$.

So, the expression for $\boldsymbol{\lambda_2}$ can now be written as

$$\boldsymbol{\lambda_2 = \{m * d * e * V / [I_x * f]\} * f_2 * CS_\alpha}$$

But, $\boldsymbol{I_x = m * d^2 * k_x^2}$

so, the expression for $\boldsymbol{\lambda_2}$ can be rewritten as

$$\boldsymbol{\lambda_2 = \{V / [f * d]\} * e * k_x^{-2} * f_2 * CS_\alpha}.$$

Right out of the muzzle

$$\boldsymbol{f = V_0 / (n * d)} \quad \textbf{revolutions/second}$$

or $\boldsymbol{n = V_0 / (f * d)}$ **calibers/turn.**

So, using ***initial values*** for each flight variable,

$$\boldsymbol{\lambda_2 = n * e * k_x^{-2} * f_2 * CS_\alpha} .$$

For the well studied 30-caliber 168-grain Sierra International bullet, for example, at an initial airspeed of Mach 2.5:

$$\mathbf{k_x^{-2} = 9.218\ calibers^{-2}}$$
$$\mathbf{I_y/I_x = 7.441}$$
$$\mathbf{CL_\alpha = 2.850}$$
$$\mathbf{CD_0 = 0.320}$$
$$\mathbf{CM_\alpha = 2.560}$$
$$\mathbf{CD_\alpha = 4.400}$$
$$\mathbf{n = 38.96\ calibers/turn\ (or\ 12\ inches/turn)}$$
$$\mathbf{f_1 + f_2 = 2800/7.441 = 376.3\ hz}$$
$$\mathbf{Sg = 1.75}$$
$$\mathbf{R = 4.79}$$
$$\mathbf{f_2 = (f_1 + f_2)/(R + 1) = 65.0\ hz}$$

And, at Mach 2.5

$$\mathbf{CS_\alpha = CM_\alpha * CD_\alpha/(CL_\alpha + CD_0)^2 = 1.121}$$

$$\mathbf{e = (k_x^2/n) * CS_\alpha = 0.002567.}$$

From data published by Robert L. McCoy of the Ballistics Research Lab (BRL) at Aberdeen Proving Ground in Maryland, the pertinent aeroballistics coefficients for this old 168-grain bullet as a function of airspeed in Mach numbers were as shown in the table below.

30-caliber 168-grain Sierra International (per McCoy)							
						(spin damp)	
	Mach No.	**CMa**	**CDa**	**CLa**	**CD0**	**Clp**	**CSa**
	2.50	2.56	4.40	2.85	0.320	-0.0068	**1.1209**
	2.20	2.69	5.40	2.68	0.339	-0.0073	**1.5937**
	2.00	2.79	6.10	2.58	0.350	-0.0075	**1.9824**
	1.80	2.88	6.80	2.45	0.365	-0.0080	**2.4714**
	1.60	2.98	7.30	2.32	0.385	-0.0083	**2.9731**
	1.40	3.06	7.60	2.15	0.410	-0.0088	**3.5486**
	1.20	3.12	6.50	1.90	0.434	-0.0095	**3.7228**
	1.10	3.15	3.60	1.70	0.447	-0.0098	**2.4601**
	1.05	3.17	3.10	1.55	0.449	-0.0099	**2.4592**
	1.00	3.24	3.00	1.35	0.430	-0.0100	**3.0678**
	0.95	3.45	2.90	1.30	0.240	-0.0103	**4.2187**
	0.90	3.43	2.90	1.35	0.160	-0.0105	**4.3625**
	0.85	3.40	2.90	1.40	0.142	-0.0107	**4.1468**
	0.80	3.38	2.90	1.45	0.140	-0.0108	**3.8772**
	0.50	3.26	2.90	1.63	0.140	-0.0125	**3.0177**
	0.00	3.05	2.90	1.75	0.140	-0.0150	**2.4761**

So, based on these aeroballistic parameters,

$$\boldsymbol{\lambda_2 = n*e*k_x^{-2}*f_2*CS_\alpha}$$

$$\boldsymbol{\lambda_2 = 59.9 \text{ seconds}^{-1}}$$

and $$\boldsymbol{\lambda_S = -\lambda_2*d/V_0 = -0.000549 \text{ calibers}^{-1}.}$$

For critical damping during each full coning cycle we would need a damping factor of

$$\boldsymbol{[\lambda_2]_{Crit} = f_2 = 65.0 \text{ sec}^{-1}}$$

So, this **λ_2** damping would be a bit less than ***critical damping*** of the coning angle **α**. This sub-critical damping of the coning angle requires only about **0.2567-percent** of the energy loss due to yaw-drag.

We can calculate a barrel twist rate **n** (calibers/turn) for just critical damping as:

$$[n]_{Crit} = [\lambda_2]_{Crit}/[e*k_x^{-2}*f_2*CS_\alpha]$$

$$[n]_{Crit} = 1/[e*k_x^{-2}*CS_\alpha]$$

$$= 37.7 \text{ calibers/turn (per Greenhill)}$$

or $$[n]_{Crit} = 37.7*(0.308 \text{ in/cal}) = 11.6 \text{ inches/turn.}$$

Unfortunately, the old 30-caliber, 168-grain Sierra International bullet was ***not*** actually dynamically stable at Mach 2.5 airspeed due to several bullet design errors. These calculations are shown as if it were stable simply because it is one of the few bullets for which we have the complete set aeroballistic coefficient data. The above formulation for $\boldsymbol{\lambda_2}$ does not apply for dynamically unstable bullets. The slow-mode damping factor for that particular bullet at Mach 2.5 was actually negative (in the formulation used herein).

For initial critical damping of the coning motion of *any* rifle bullets which are dynamically stable, we can formulate the barrel twist-rate required $\mathbf{[n]_{Crit}}$ in **calibers/turn**. Since $\mathbf{[\lambda_2]_{Crit} = f_2}$, the expression for $\mathbf{[n]_{Crit}}$ reduces to:

$$[n]_{Crit} = 1/(e*k_x^{-2}*CS_\alpha)$$

with $$CS_\alpha = CM_\alpha*CD_\alpha/(CL_\alpha + CD_0)^2$$

and all coefficients evaluated at muzzle speed.

With the constant fraction $\mathbf{e = 0.0023345}$ and $\mathbf{k_x^{-2} \approx 9.0\ calibers^{-2}}$ for monolithic VLD and ULD rifle bullets, the expression for $\mathbf{[n]_{Crit}}$ becomes

$$[n]_{Crit} = 47.6 \text{ calibers}/CS_\alpha.$$

As shown in the table above for the old Sierra International bullet, the Damping Coefficient $\mathbf{CS_\alpha}$ varied for different muzzle speeds from **1.121** at Mach 2.5 up to **3.723** at Mach 1.20.

For long-nosed monolithic copper-alloy ULD bullets, we can expect $\mathbf{CD_\alpha}$ to be larger and $\mathbf{(CL_\alpha + CD_0)}$ to be smaller at high Mach-speeds out of the muzzle, so expecting an initial value of about **2.0** to **2.5** for **CS_α** is not unreasonable for these modern bullets.

As the Damping Coefficient **CS_α** increases, the barrel twist-rate required for critical damping of the coning angle **α**, and thus for achieving early hyper-stable bullet flight, must get "faster." That is, **$[n]_{Crit}$** in **calibers/turn** must get smaller.

For those firing monolithic copper-alloy bullets at Mach 3.0 to Mach 3.5, the single best recommended barrel twist-rate should be:

$$[n]_{Crit} \approx 20 \text{ calibers/turn.}$$

By ensuring critical damping of the coning angle initially, a bullet fired from a barrel having **19** to **21 calibers per** turn twist-rate and entering the undisturbed ambient atmosphere a few yards ahead of the rifle with **zero** yaw attitude and **zero** yaw-rate should achieve hyper-stability initially and maintain it throughout its flight to an extremely long-range (ELR) target. This copper-alloy ULD bullet would be flying with minimum aerodynamic drag due only to its designer-minimized zero-yaw coefficient of drag **CD_0** all the way to its maximum-range target.

The initial gyroscopic stability **Sg** of such a monolithic ULD bullet fired from a barrel having this critical twist-rate should be approximately **3.0 (R = 10)**. The initial dynamic stability for these bullets should then be **0.33 (R = 10)**. These bullets should then be exceedingly stable in transiting the turbulent transonic speed region far downrange and should then continue flying with minimum yaw (coning angle) as reasonably good subsonic bullets.