

Thoughts on Dynamic Stability of Rifle Bullets

March 24, 2018

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General

Gyroscopic Stability (Sg) refers to the ability of a spin-stabilized, statically unstable rifle bullet to resist tumbling in flight caused by its aerodynamic overturning moment. This overturning moment causes the spinning bullet, acting as a gyroscope, to precess and nutate in free flight instead of tumbling erratically. These two gyroscopic reactions cause the spin-axis of the bullet to trace out its familiar epicyclic motions in pitch and yaw attitude angles. The slow-mode motion is gyroscopic precession, and the fast-mode motion is gyroscopic nutation. The slow-mode motion is also called “coning motion.”

The angular **rates** of these fast-mode and slow-mode spin-axis cyclic motions are ω_1 and ω_2 , respectively, in radians per second. They are readily found from the Tri-Cyclic Theory relationships:

$$\omega_1 + \omega_2 = (I_x/I_y) * \omega$$

$$R = \omega_1/\omega_2$$

$$Sg = (R + 1)^2/(4*R)$$

where ω is the instantaneous spin-rate of the rifle bullet in radians per second and I_x/I_y is the ratio of the second moments of inertia about crossed principal axes for the mass distribution of that bullet. The stability ratio **R** is a better, more sensitive, indicator of gyroscopic stability than is **Sg** itself.

Sg is classically defined in aeroballistics as

$$Sg = P^2/(4*M) = (\omega_1 + \omega_2)^2/(4*\omega_1*\omega_2) = (R + 1)^2/(4*R)$$

Dynamic stability (Sd) deals with the **rates of change** in the angular **amplitudes** of the coning and nutation motions of the spin-axis of the bullet in flight. The slow-mode “coning motion” of the CG of the spinning bullet around the mean trajectory is a type of **damped harmonic motion**. If the

coning angle α , for example, decreases in size with ongoing flight time t , that motion is said to be “damped.” If $\alpha(t)$ increases in size or remains constant with ongoing time-of-flight, it is called an “undamped” motion. Ballisticians have long modeled this damping as exponential in downrange flight distance (s) measured in projectile calibers (d) with fast-mode and slow-mode damping factors, λ_F and λ_S , respectively, in inverse calibers.

Here, the fast-mode and slow-mode exponential damping factors are λ_1 and λ_2 , respectively, in inverse seconds, such that, for the slow-mode coning angle α ,

$$\alpha(t) = \alpha(0) * \exp[-\lambda_2 * t] .$$

The damping with λ_1 of the fast-mode is modeled similarly, but because that nutation damping seems never to be a problem in rifle shooting, we shall ignore it for a time here.

Note that in accordance with modern engineering practice, we have reversed the signs of the damping factors λ_1 and λ_2 from those used in classic aeroballistics (λ_F and λ_S , respectively), and also they are now per unit of time t (in seconds) rather than per calibers d of path distance s travelled. Thus, when $\lambda_2 > 0$, the coning motion is damped, and for $\lambda_2 \leq 0$, the coning angle is undamped,

$$\begin{aligned} \text{and} \quad \lambda_1 &= -\lambda_F * V/d \text{ (in inverse seconds)} \\ \lambda_2 &= -\lambda_S * V/d \text{ (in inverse seconds).} \end{aligned}$$

where V = the instantaneous airspeed of the bullet.

If either damping factor, λ_1 or λ_2 , goes negative and remains there for a significant time during a flight, the angular magnitude of either the fast-mode nutation or the coning motion will increase without bound. When, for example, the coning angle α approaches **90 degrees**, we would certainly call that a failure in dynamic stability.

Yaw-drag additionally retards the forward motion of the bullet due to its flying with the coning angle α as its long-term aerodynamic angle-of-attack. For a dynamically stable rifle bullet, some small fraction e of its kinetic energy of forward motion is continuously being bled off to reduce the amplitude α of that bullet’s coning motion. In the absence of fast-mode nutation, the fraction e of that yaw-drag frictional energy loss goes toward

“damping” the slow-mode coning motion. Another small fraction **k** of the bullet’s kinetic energy is also used to re-orient the axis of the coning motion in response to any step-change in the direction of approach of the apparent wind experienced by the flying bullet.

Since the CG of even a marginally stable bullet shows no measurable motion at the fast-mode nutation rate, there is almost no kinetic energy associated with that type of gyroscopic motion of the bullet’s spin-axis direction and any associated “fast-mode coning motion” of the CG.

As kinetic energy is extracted from the orbital motion of the CG around its mean trajectory at the gyroscopic precession (or coning) rate, the radius **r** of that orbit must decrease correspondingly to a lower orbital potential energy state, along with its associated coning angle **α**, with **$r = D \cdot \sin(\alpha)$** . This is the frictional damping mechanism which reduces the coning angle **α** over the distance **s** traversed by the bullet along the mean trajectory or, alternatively, over its time-of-flight **t**.

Aerodynamic drag itself is a frictional force in that it can only act to oppose or retard bullet motion through the air. The total decrease in the bullet’s kinetic energy due to this retardation of forward motion is caused only by the total force of aerodynamic drag **F_D** experienced by that bullet at any time during its flight. The deceleration of the bullet is **F_D/m** at any point during its free flight. Except for damping or re-orienting the coning motion and nutation of the bullet, the remaining frictional energy loss is dissipated as heat energy. We shall explore this energy relationship below.

Gyroscopic Stability

Various estimators for calculating the required rifling twist-rates for adequate gyroscopic stability **Sg** for rifle bullets have been used since the Greenhill Formula of the mid-1800's. We also now use Don Miller's formulation for VLD-type bullets and Bob McCoy's McGYRO calculations developed for artillery projectiles. A common feature of these estimators is that they all rely heavily upon the bullet length **L** in calibers as a type of slenderness ratio of the projectile. Some formulations also adjust for muzzle velocity, air density, and the average material density of the projectile.

Ideas are changing about what values of **Sg** constitute adequate and desired initial gyroscopic stability, especially in the flat-firing of rifles to extended ranges. Formerly, we considered an initial **Sg** of **1.2** to **1.4** to be adequate for best short-range rifle accuracy. Now, we realize the advantages of reduced aerodynamic drag if we provide our conventional jacketed, lead-cored rifle bullets with an initial **Sg** of at least **1.5**, as recommended by Bob McCoy and Bryan Litz. Rifleman are now learning to launch the new monolithic copper-alloy ultra-low-drag (ULD) bullets with an initial **Sg** of **2.5** to **3.0** for best results in extreme long-range (ELR) shooting. This requires a rifling twist-rate of about **20 calibers per turn** for monolithic ULD bullets of about **5.5 calibers** in length **L**. The resulting bullet spin-rates at high muzzle speeds are not compatible with use of conventional jacketed lead-cored match bullets.

New Analytical Calculations

Munk's Equation from early aerodynamics allows us to estimate the overturning moment **M** acting upon a slender solid-body-of-revolution in a laminar flow-field. Munk was a student of Prandtl in Germany in the early 20th Century. In our terms, Munk's Equation is

$$\mathbf{M} = \mathbf{q} * \mathbf{Sin}(2 * \alpha) * (\mathbf{Vol} - \mathbf{S} * \mathbf{Xcg})$$

where

Vol = Volume of projectile

Xcg = Distance from nose to CG.

The doubling of the aerodynamic angle-of-attack α comes directly from observations of telldales for solids-of-revolution in laminar flow-fields. ULD rifle bullets fly with laminar boundary layer flow-fields over their ogives at supersonic and subsonic airspeeds.

A paper published by Ing. Dr. B. Kneubuehl of Thun, Switzerland, entitled “What is the maximum length of a spin stabilized projectile,” details an analytical procedure for calculating the important aeroballistic parameters of I_y/I_x ratio, CM_α , and initial Sg from basic data for any reasonable projectile having a homogeneous mass distribution. I discovered this little gem of ballistics papers on Research Gate.

For a simple cylinder-and-cone projectile model, we can analytically calculate the mass properties and, from Munk’s Equation, the aeroballistic moment coefficient CM_α of the projectile:

$$Wt(calc) = (\pi/4) * \rho_p * d^3 * L * (1 - 2*h/3) \quad \text{grains}$$

$$I_x = (\pi/32) * \rho_p * d^5 * L * (1 - 4*h/5) \quad \text{grain-inches}^2$$

$$I_y = (\pi/960) * \rho_p * d^5 * L * f_1(L, h) \quad \text{grain-inches}^2$$

$$I_y/I_x = f_1(L, h) / [30 * (1 - 4*h/5)]$$

$$CM_\alpha = L * f_2(h)$$

where $Wt(calc)$ = Calculated weight of projectile in grains
 d = Reference diameter of bullet in inches = 1.0 caliber
 ρ_p = Density of monolithic bullet in grains/cubic inch
 L = Bullet length in calibers
 $h = L(nose)/L$

$$f_1(L, h) = 15 - 12*h + L^2 * (60 - 160*h + 180*h^2 - 96*h^3 + 19*h^4) / (3 - 2*h)$$

and $f_2(h) = (18 - 24*h + 7*h^2) / (18 - 12*h).$

The initial gyroscopic stability Sg of this cylinder-cone projectile model can then also be analytically calculated as

$$Sg = 0.300 * (\rho_p / \rho) * [\tan^2(180/n)] * (5 - 4*h)^2 / [f_1(L, h) * f_2(h)]$$

where ρ = Ambient air density
 n = Rifling twist-rate in calibers/turn.

The angular argument of the tangent function is just the helix angle of the rifling twist of the barrel given by $180/n$ in degrees.

Adjusting the Swiss Formulation for ULD Rifle Bullets

The Swiss paper is primarily concerned with artillery shell ballistics. We have enough data from designing solid monolithic ULD rifle bullets to customize these analytical calculations for any similar rifle bullets constructed with homogeneous mass density.

We can adjust $f_2(h)$ by a factor of **2.75/3.60** to bring the analytically calculated overturning moment coefficient CM_α into agreement with wind-tunnel data measurements for rifle bullet models, explicitly at Mach 2.5, but somewhat applicable for all supersonic airspeeds:

$$f_{2A}(h) = (2.75/3.60) * f_2(h).$$

For a given rifle bullet, the aeroballistic overturning moment coefficient CM_α tends to be approximately the same value for all supersonic and subsonic airspeeds, but varies up and down during transonic instability.

The actual weight **Wt** of the a solid monolithic ULD rifle bullet is **1.1078 times** the analytically calculated weight **Wt(calc)** for the corresponding cylinder-cone model of that rifle bullet. The ULD bullet design utilizes either a secant ogive with **RT/R = 0.500** or a Sears-Haack minimum drag nose shape and also features an aerodynamically effective boat-tail of about **0.7 calibers** in length. Since we now know both of these weights, we can use their ratio **Wt/Wt(calc)** to adjust the analytically calculated ratio of second moments **Iy/Ix**:

$$\{Iy/Ix\}_A = 1.1078^{1.11} * f_1(L,h)/[30*(1 - 4*h/5)]$$

$$\{Iy/Ix\}_A = 1.12035 * f_1(L,h)/[30*(1 - 4*h/5)].$$

This adjustment brings these analytical calculations of **Iy/Ix** into agreement with our numerical integrations of these mass properties for a wide array of different solid monolithic ULD rifle bullets.

We associate this weight-ratio adjustment with $f_1(L,h)$, and adjust that analytical function accordingly:

$$f_{1A}(L,h) = 1.12035 * f_1(L,h)$$

so that

$$\{I_y/I_x\}_A = f_{1A}(L,h)/[30*(1 - 4*h/5)].$$

We can then separately adjust the calculated second moment of inertia I_y by the factor

$$1.1078^{2.02} = 1.22974$$

and the calculated value for I_x by the factor

$$1.1078^{0.91} = 1.09764$$

so that

$$\{I_y\}_A = 1.22974 * (\pi/960) * \rho_p * 0.3302^5 * L * f_{1A}(L,h)$$

$$\{I_x\}_A = 1.09764 * (\pi/32) * \rho_p * 0.3302^5 * L * (1 - 4*h/5)$$

with both second moments of inertia given in **grain-inches²**.

Note that

$$1.1078^{2.02}/1.1078^{0.91} = 1.1078^{1.11}$$

so the ratio I_y/I_x remains as previously adjusted.

Neither I_y nor I_x should separately scale with the fifth power of caliber d^5 for rifle bullets of different sizes. We worked out these weight-ratio adjustments for a 338-caliber monolithic ULD bullet having a reference diameter d of **0.3302 inches**. So, $d = 0.3302$ inches must be used in calculating the adjusted values of I_y and I_x for all other calibers of similar rifle bullets.

These adjusted analytical calculations of the second moments of inertia, I_y and I_x , then agree closely with corresponding numerically integrated values for solid monolithic ULD rifle bullets of all calibers.

Finally, we can get good agreement with Miller and McGYRO initial S_g estimates for solid monolithic ULD rifle bullets if we replace the initial constant factor of **0.300** in the S_g formulation with **$0.2254 * (0.410/d)^{0.125}$** , where d is that bullet's reference diameter in inches. This formulation is

optimized for rifle bullets of 30- to 50-caliber. The adjusted analytic formulation of initial **Sg** then becomes

$$\{\mathbf{Sg}\}_A = 0.2254 * (0.410/d)^{0.125} * (\rho_p/\rho) * [(5 - 4*h)*\text{Tan}(180/n)]^2 / [f_{1A}(L,h) * f_{2A}(h)]$$

Both the air density ρ and projectile density ρ_p are used explicitly here, but no separate correction for muzzle velocity variations is available. This formulation calculates only the initial **Sg**, and does not work later in the flight.

Dynamic Stability

In classic aeroballistics, the dynamic stability of a spin-stabilized projectile is given as

$$S_d = 2T/H = 2(PT)/(PH)$$

$$S_d = 2(\omega_1\lambda_2 + \omega_2\lambda_1)/[(\omega_1 + \omega_2)(\lambda_1 + \lambda_2)]$$

The conditions for simultaneous gyroscopic and dynamic stability are given as

$$S_g > 1/[S_d(2 - S_d)] > 1$$

and

$$0 < S_d < 2$$

As **S_d** approaches **0**, the slow-mode (coning) motion becomes less damped, and as **S_d** approaches **2**, the fast-mode (nutation) motion becomes less damped.

Noting the symmetry about **S_d = 1**, we will formulate an new expression for **(S_d – 1)** which is symmetric about **zero**. After some algebra, we find:

$$S_d - 1 = [(\omega_1 - \omega_2)/(\omega_1 + \omega_2)][(\lambda_2 - \lambda_1)/(\lambda_2 + \lambda_1)]$$

$$S_d - 1 = [(R - 1)/(R + 1)][(\lambda_2 - \lambda_1)/(\lambda_2 + \lambda_1)]$$

with

$$-1 < (S_d - 1) < +1 \quad (\text{for dynamic stability}).$$

We know that $\omega_1 > \omega_2$ because $\omega_1 = \omega_2$ is the boundary condition for minimum gyroscopic stability where **S_g = R = 1**.

Now suppose that the damping factors are such that each type of angular motion is reduced by the *same damping fraction* during each cycle of its motion. In this case, we would have

$$\lambda_1/\omega_1 = \lambda_2/\omega_2$$

or

$$\lambda_1 = (\omega_1/\omega_2)*\lambda_2 = R*\lambda_2$$

Then, after more algebra

$$S_d - 1 = - [(R - 1)/(R + 1)]^2$$

Or

$$S_d = 1 - [(R - 1)/(R + 1)]^2$$

In this special case, the formulation shows that

$$0 < S_d < 1$$

whenever the gyroscopic stability ratio $R > 1$, and the fast-mode motion is always certainly well damped.

The rifle bullet motions observed in 6-degree-of-freedom flight simulations often tend to behave in just about this way, and there might even be sound physical reasons behind this type of behavior. With initial R -values of about **6** to **10** for rifle bullets which are gyroscopically well stabilized with initial S_g from **1.8** to **3.0**, the fast-mode cycles occur just R -times more frequently over time than do the coning cycles. With the λ_1 fast-mode damping factor also R -times larger than λ_2 , the nutations would damp to insignificance R -times more quickly over time than does the coning motion in this special case. No lack of adequate fast-mode damping seems ever to be observed for rifle bullets fired with adequate initial gyroscopic stability.

The table included below shows this special case relationship for values of R from **1** to **37**, where $S_g = 9.757$. Note that the conditions for both gyroscopic and dynamic stability are continually met in this special case. As R continually increases during the flight, the dynamic stability S_d decreases toward **0**. This indicates that the amplitude of the slow-mode coning motion $\alpha(t)$, while becoming less and less damped as R increases, is never completely undamped in this special case.

For Constant Damping Ratio per Cycle:

$$\lambda_1/\lambda_2 = \omega_1/\omega_2$$

$$S_d = 1 - ((R - 1)/(R + 1))^2$$

<u>S_g</u>	<u>R</u>	<u>S_d</u>	<u>$1/(S_d*(2-S_d))$</u>	<u>$1/S_g <$</u>	<u>$S_d*(2-S_d)$</u>
1.000	1.000	1.0000	1.0000	1.0000	1.0000
1.125	2.000	0.8889	1.0125	0.8889	0.9877
1.333	3.000	0.7500	1.0667	0.7500	0.9375
1.563	4.000	0.6400	1.1489	0.6400	0.8704
1.800	5.000	0.5556	1.2462	0.5556	0.8025
2.042	6.000	0.4898	1.3519	0.4898	0.7397
2.286	7.000	0.4375	1.4629	0.4375	0.6836

2.531	8.000	0.3951	1.5772	0.3951	0.6340
2.778	9.000	0.3600	1.6938	0.3600	0.5904
3.025	10.000	0.3306	1.8120	0.3306	0.5519
3.273	11.000	0.3056	1.9314	0.3056	0.5177
3.521	12.000	0.2840	2.0518	0.2840	0.4874
3.769	13.000	0.2653	2.1729	0.2653	0.4602
4.018	14.000	0.2489	2.2945	0.2489	0.4358
4.267	15.000	0.2344	2.4165	0.2344	0.4138
4.516	16.000	0.2215	2.5389	0.2215	0.3939
4.765	17.000	0.2099	2.6617	0.2099	0.3757
5.014	18.000	0.1994	2.7846	0.1994	0.3591
5.263	19.000	0.1900	2.9078	0.1900	0.3439
5.513	20.000	0.1814	3.0312	0.1814	0.3299
5.762	21.000	0.1736	3.1547	0.1736	0.3170
6.011	22.000	0.1664	3.2784	0.1664	0.3050
6.261	23.000	0.1597	3.4021	0.1597	0.2939
6.510	24.000	0.1536	3.5260	0.1536	0.2836
6.760	25.000	0.1479	3.6500	0.1479	0.2740
7.010	26.000	0.1427	3.7740	0.1427	0.2650
7.259	27.000	0.1378	3.8981	0.1378	0.2565
7.509	28.000	0.1332	4.0223	0.1332	0.2486
7.759	29.000	0.1289	4.1465	0.1289	0.2412
8.008	30.000	0.1249	4.2708	0.1249	0.2341
8.258	31.000	0.1211	4.3951	0.1211	0.2275
8.508	32.000	0.1175	4.5195	0.1175	0.2213
8.758	33.000	0.1142	4.6439	0.1142	0.2153
9.007	34.000	0.1110	4.7684	0.1110	0.2097
9.257	35.000	0.1080	4.8928	0.1080	0.2044
9.507	36.000	0.1052	5.0174	0.1052	0.1993
9.757	37.000	0.1025	5.1419	0.1025	0.1945

Achieving Minimum Coning Angle Flight

David Tubb launched our new 225-grain copper 338-caliber ULD bullets in his 1000-yard testing with an initial **R**-value of **8.86 (Sg = 2.75)** from his 7.5-inch twist Schneider 338 barrel. The values of **R** and **Sg** could only have increased during their supersonic flight to his 1000 yard target. The resulting rather high ballistic coefficient **BC(G1)** measurement of **0.794** for an average airspeed of **Mach 2.46** indicates that these bullets were flying

with a “minimum coning angle” motion ($\alpha \approx \Delta\Phi \ll 0.1$ degree) during most of their 1000-yard flights and certainly that they were dynamically stable ($\lambda_2 > 0$) all the way to that target distance. David was able to measure the aerodynamic drag of these test bullets due solely to their zero-yaw coefficient of drag CD_0 . I propose that we term this flight mode “hyper-stable flight.” Otherwise, the bullet drag measurements from the usual firing tests contain significant yaw-drag contributions. The bullet’s actual coning angle-of-attack α is difficult and expensive to measure in flight.

In *minimum coning angle flight*, the increase $\Delta\Phi$ in coning angle α due to the gravitational downward curving of the mean trajectory during each half coning cycle is closely matched by the exponential slow-mode damping of the coning angle α during that half coning cycle:

$$\alpha \leq (\alpha + \Delta\Phi) \cdot \exp[-\lambda_2 \cdot \pi / \omega_2]$$

This condition eventually produces a steady-state coning angle of $\alpha > \Delta\Phi \ll 0.1$ degree whenever $\lambda_2 \geq [\lambda_2]_{\text{Min}}$ where:

$$[\lambda_2]_{\text{Min}} = (\omega_2 / \pi) \cdot \ln[(\alpha + \Delta\Phi) / \alpha]$$

or $[\lambda_2]_{\text{Min}} = 2 \cdot f_2 \cdot \ln[(\alpha + \Delta\Phi) / \alpha]$.

As the damping coning angle α approaches as its lower limit, the change in flight path angle $\Delta\Phi$ during this half coning cycle,

$$\ln[(\alpha + \Delta\Phi) / \alpha] \leq \ln[2] = 0.693147.$$

Then, for the 225-grain bullet test-fired, for which the initial coning frequency f_2 is given by

$$f_2 = [(12 \cdot 3378 \text{ fps} / 7.5 \text{ in.}) / 12.1227] / 9.86 = 45.22 \text{ hz}$$

where $ly/lx = 12.1227$

$$R + 1 = 9.86.$$

Then $[\lambda_2]_{\text{Min}} \leq 2 \cdot f_2 \cdot \ln(2) = 62.69 \text{ seconds}^{-1}$

If $\lambda_2 = [\lambda_2]_{\text{Min}}$, the time constant for damping reduction of the size of the coning angle α by a factor of $1/e = 0.3679$ is **15.95 milliseconds**, or **0.7213 coning cycles** (or **53.9 feet** of early flight).

An under-damped driven linear system continues oscillating at its driving frequency, but at a reduced amplitude. An over-damped driven linear system stops oscillating at that frequency, but never quite achieves equilibrium either. A critically damped linear system stops its oscillation as quickly as possible.

Here, we are dealing with a harmonic 2-dimensional rotational motion of the CG of the coning bullet. Only the radial angular magnitude α of this coning motion is being frictionally damped with ongoing time-of-flight or flight distance. The rate of the coning motion is completely independent of its amplitude. Thus, the linear system damping concepts do not fully apply here.

But is this damping really independent of the orbital velocity v of the CG of the bullet?

$$v = r \cdot \omega_2 = D \cdot \sin(\alpha) \cdot \omega_2 \approx \alpha \cdot D \cdot \omega_2$$

The coning bullet is moving laterally (sideways) through the air at a subsonic airspeed v and presenting its largest possible cross-sectional area to that airflow. The frontal cross-sectional area is

$$S = (\pi/4) \cdot d^2$$

If a rifle bullet has a volume **Vol** of about **3.0 cubic calibers**, as with many monolithic VLD and ULD bullets, its minimum side aspect area **Sa** would be **La*d**, where **La** is given by

$$La = Vol/S = 3.0 \cdot d^3 / [(\pi/4) \cdot d^2] = (12/\pi) \cdot d = 3.820 \cdot d$$

and

$$Sa = La \cdot d = 3.820 \cdot d^2 = 4.863 \cdot S.$$

For bullets **5.5 calibers** in length, **Sa** is probably more like **5.5*S**.

If the subsonic coefficient of drag **CD** for this rifle bullet is about **0.100**, the drag force **F_{DC}** due to this coning motion would be

$$F_{DC} = (\rho/2) \cdot v^2 \cdot Sa \cdot CD = (\rho/2) \cdot \alpha^2 \cdot (D \cdot \omega_2)^2 \cdot 0.4863 \cdot S$$

The minimum orbital kinetic energy loss per half coning cycle **ΔE_c** would be

$$\Delta E_c = F_{DC} \cdot \pi \cdot r = \pi \cdot (\rho/2) \cdot \alpha^3 \cdot D^3 \cdot (\omega_2)^2 \cdot 0.4863 \cdot S$$

But, as will be shown later, the loss in orbital potential energy with α per half coning cycle is

$$\Delta E_c = [m(D\omega_2)^2]\alpha\Delta\alpha.$$

Setting these energy losses equal, we have

$$\Delta\alpha = \alpha^2 * (\pi/m) * (\rho/2) * D * 0.4863 * S$$

$$\Delta\alpha = \alpha * [1 - \exp(-\lambda_c * t_2/2)]$$

$$\exp(-\lambda_c * t_2/2) = 1 - \alpha * [(\pi/m) * (\rho/2) * D * 0.4863 * S]$$

$$\lambda_c = (-2 * f_2) * \ln\{1 - \alpha * [0.764 * (\rho/m) * D * S]\}.$$

$$\lambda_c = (-2 * f_2) * \ln\{1 - \alpha * [0.600 * (\rho/m) * d^2 * D]\}.$$

For a **250-grain 338 bullet** in an ICAO sea-level atmosphere ($\rho = 0.0764742 \text{ lbm/ft}^3$), with $f_2 = 45 \text{ hertz}$, $D = 4 * d$, and $\alpha = 0.100 \text{ radians}$,

$$\lambda_c = (-2 * f_2) * \ln\{1 - \alpha * [0.600 * (\rho/m) * D * d^2]\}.$$

$$\lambda_c = -90 \text{ hz} * \ln[1 - 0.100 * [0.600 * 2.1413 * 4 * (0.028167)^3]]$$

$$\lambda_c = -90 \text{ hz} * \ln[1 - 0.000011484]$$

$$\lambda_c = -90 \text{ hz} * [-0.000011484] = 0.0010336 \text{ seconds}^{-1}.$$

Critical damping would have a time constant of **967.5 seconds** at this tiny λ_c damping rate.

Since we are considering flight times of only a few seconds, we should be justified in saying that the damping factor λ_2 affects only the angular size α of the coning motion and not the orbital motion of the CG.

Minimum coning angle flight is achieved earlier in the bullet's flight when the bullet is perfectly launched with zero initial yaw and yaw-rate, when the initial spin-rate of the bullet is very high, when the bullet design is easier to stabilize gyroscopically (**$S_g \geq 2.5$**), when crosswinds are light and steady, when the density of the ambient atmosphere is relatively low, and when bullet velocity is very high.

Importantly, the coning motion of the bullet during this “minimum coning angle” hyper-stable flight mode still allows the rotational cancellation of the aerodynamic lift force acting on the bullet due to any crosswinds. Windage corrections would have to be at least an order of magnitude greater if this were not the case. Windage corrections remain attributable only to the aerodynamic drag force as first formulated by Dedion in 1859.

Ordinary outdoor test-firing for most bullets allows measurement of a total aerodynamic drag force which includes a significant yaw-drag component due to coning angles-of-attack often in the **2 to 5-degree** range. This coning angle is effectively a long-term aerodynamic angle-of-attack, but these small attitude angles are difficult and expensive to measure, especially in outdoor firing tests. We understand that measured aerodynamic drag is reduced somewhat merely by increasing the fired bullet’s initial gyroscopic stability from a marginal **Sg** of **1.4** to a nominal **Sg** of **1.5**. It stands to reason that increasing the initial **Sg** a lot further might decrease measured aerodynamic drag even more.

Hyper-stabilizing our bullets with an initial **Sg** of **2.75** in these test-firings effectively allowed their pure **CD₀** aerodynamic drag coefficient to be measured for the average Mach-speed over the entire flight to the target (**Mach 2.46**). However, increasing initial **Sg** even further should *not* be expected to provide very much (if any) additional reduction in measured aerodynamic drag below the **CD₀** coefficient for zero-yaw flight.

Energy Considerations

The total energy **TE** of the fired rifle bullet which is conserved in ballistic flight is

$$\mathbf{TE} = \mathbf{E} + \mathbf{E_c} + \mathbf{P_c} + \mathbf{mgh} + \mathbf{Heat}$$

where

E = Kinetic energy of the bullet due to forward motion

E_c = Kinetic energy of the coning motion

P_c = Potential energy of the coning motion

mgh = Gravitational potential energy of bullet

Heat = Dissipated energy absorbed by surroundings.

Here, we are only interested in the first three energy terms.

We can gain additional insight into this steady-state “minimum coning angle” hyper-stable flight by looking at the bullet’s loss of kinetic energy **E** in flight. Let us say that at any time **t** during flight, the loss in kinetic energy **ΔE** over the small time interval **Δt** is governed by the Equations of Motion as:

$$E(t + \Delta t) = E(t) - \Delta E(\Delta t)$$

and

$$\Delta E(\Delta t) = F_D * \Delta s = F_D * V * \Delta t$$

where **F_D** is the total aerodynamic force of drag and **Δs** is the path length (in feet) travelled during the small time-interval **Δt**.

In particular, we are interested in the loss in kinetic energy **ΔE** during any particular half coning cycle where

$$\Delta t = (2\pi/\omega_2)/2 = 1/(2*f_2) = T_2/2 \text{ seconds.}$$

One half of the period of the coning motion is the time interval during which the coning motion adjusts to any step-change in the direction of the apparent wind approaching the flying bullet.

In linear aeroballistics theory, **F_D** is accurately modelled as

$$F_D = q * S * (CD_0 + \delta^2 * CD_\alpha)$$

where

$$\delta = \text{Sin}(\alpha) \approx \alpha.$$

CD₀ is the coefficient of minimum drag for exactly nose-forward aerodynamic flight at a given airspeed (Mach Number), and **CD_α** is the yaw-drag coefficient at that same airspeed.

Now the expression for kinetic energy loss in any particular half coning cycle becomes

$$\Delta E(T_2/2) = q * S * (CD_0 + \alpha^2 * CD_\alpha) * V * T_2/2$$

The sensitivity of this expression to coning angle **α** is given by its partial derivative with respect to **α** :

$$\partial(\Delta E)/\partial \alpha = 2 * \alpha * q * S * V * CD_\alpha * T_2/2$$

$$\partial(\Delta E)/\partial \alpha = \alpha * q * S * V * CD_\alpha / f_2.$$

Coning Kinetic Energy

We can also formulate the kinetic energy E_c of the orbital coning motion itself as

$$E_c = (m/2) * (r * \omega_2)^2 = (m/2) * (D * \sin \alpha * \omega_2)^2$$

$$E_c = (m/2) * (D * \omega_2)^2 * \alpha^2$$

where D is the slowly varying coning distance of the CG of the bullet from its coning apex. Here we are again using the small angle approximation for small coning angles α .

Forming the partial derivative again with respect to α , we find the sensitivity of E_c to α to be

$$\partial(E_c)/\partial\alpha = m * (D * \omega_2)^2 * \alpha.$$

We now reason that the kinetic energy ΔE_c of the coning motion lost to frictional damping of the coning angle by the difference $\Delta\alpha$ during each of these “steady-state” half coning cycles must be a small fraction e of the kinetic energy ΔE extracted from the forward motion of the bullet due to that same coning angle difference $\Delta\alpha$ during that same half coning cycle.

For $(\Delta\alpha, \alpha) \neq 0$, we can write

$$e * \Delta\alpha * \partial(\Delta E)/\partial\alpha = \Delta\alpha * \partial(E_c)/\partial\alpha$$

and

$$e * (\alpha^2 * 2\pi * q * S * V / \omega_2) * C_{D\alpha} = \alpha * m * (D * \omega_2)^2$$

$$e * (2\pi * q * S * V / \omega_2) * C_{D\alpha} = m * (D * \omega_2)^2$$

From Coning Theory, we know that the distance D (in feet) is given by

$$D = q * S * (C_{L\alpha} + C_{D0}) / [m * (\omega_2)^2]$$

or

$$D * \omega_2 = q * S * (C_{L\alpha} + C_{D0}) / [m * \omega_2].$$

Substituting for $D * \omega_2$ and simplifying, we have

$$e * (2\pi * q * S * V / \omega_2) * C_{D\alpha} = m * (q * S)^2 * (C_{L\alpha} + C_{D0})^2 / [m * \omega_2]^2$$

or
$$e \cdot (2\pi \cdot m \cdot V / \omega_2) \cdot CD_\alpha = (q \cdot S) \cdot (CL_\alpha + CD_0)^2 / [\omega_2]^2$$

and
$$e \cdot (2\pi \cdot m \cdot V \cdot \omega_2) \cdot CD_\alpha = (q \cdot S) \cdot (CL_\alpha + CD_0)^2$$

Also, from Coning Theory, we know that the magnitude of the coning rate ω_2 is given by

$$\omega_2 = q \cdot S \cdot d \cdot CM_\alpha / (I_x \cdot \omega)$$

where
$$I_x = m \cdot d^2 \cdot k_x^2$$

and k_x = Radius of Gyration of the bullet's mass distribution about its x -axis given in units of calibers (d , in feet/caliber).

Substituting for ω_2 and simplifying, we have

$$(q \cdot S \cdot d \cdot CM_\alpha) \cdot (e \cdot 2\pi \cdot m \cdot V \cdot CD_\alpha) = q \cdot S \cdot (CL_\alpha + CD_0)^2 \cdot (\omega \cdot m \cdot d^2 \cdot k_x^2)$$

$$e \cdot 2\pi \cdot V \cdot CM_\alpha \cdot CD_\alpha = (\omega \cdot d \cdot k_x^2) \cdot (CL_\alpha + CD_0)^2$$

or
$$(\omega \cdot d \cdot k_x^2) / (e \cdot 2\pi \cdot V) = CM_\alpha \cdot CD_\alpha / (CL_\alpha + CD_0)^2.$$

But, the auxiliary parameter P given in radians of bullet rotation per caliber of bullet travel in classic aeroballistics is given by **Eq. 50** as

$$P = (I_x / I_y) \cdot p \cdot d / V = (k_x / k_y)^2 \cdot \omega \cdot d / V$$

So,

$$(\omega \cdot d \cdot k_x^2) / (e \cdot 2\pi \cdot V) = [k_y^2 / (e \cdot 2\pi)] \cdot P$$

And,

$$[k_y^2 / (e \cdot 2\pi)] \cdot P = CM_\alpha \cdot CD_\alpha / (CL_\alpha + CD_0)^2$$

Or,

$$P = e \cdot 2\pi \cdot k_y^{-2} \cdot CM_\alpha \cdot CD_\alpha / (CL_\alpha + CD_0)^2.$$

And,

$$\omega \cdot d / V = e \cdot 2\pi \cdot k_x^{-2} \cdot [k_y^2 / (e \cdot 2\pi)] \cdot P = (I_y / I_x) \cdot P$$

$$\omega * d / V = e * 2\pi * k_x^{-2} * CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2$$

$$\omega = e * 2\pi * k_x^{-2} * (V/d) * CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2$$

In particular, right out of the muzzle at $t = 0$,

$$\omega_0 = 2\pi * V_0 / Tw = 2\pi * V_0 / (n * d)$$

So,

$$\omega_0 * d / V_0 = 2\pi / n = (I_y / I_x) * P_0$$

And,

$$n = 2\pi / [(I_y / I_x) * P_0] = 2\pi * (k_y / k_x)^{-2} / [e * 2\pi * k_y^{-2} * CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2]$$

$$n = (k_x^2 / e) * (CL_\alpha + CD_0)^2 / (CM_\alpha * CD_\alpha)$$

or

$$e = (k_x^2 / n) * (CL_\alpha + CD_0)^2 / (CM_\alpha * CD_\alpha).$$

For the well studied 30-caliber 168-grain Sierra International bullet, for example, at an initial airspeed of Mach 2.5:

$$\begin{aligned} L &= 3.98 \text{ calibers} \\ k_x^{-2} &= 9.218 \text{ calibers}^{-2} \\ CL_\alpha &= 2.850 \\ CD_0 &= 0.320 \\ CM_\alpha &= 2.560 \\ CD_\alpha &= 4.400 \end{aligned}$$

Greenhill's formula suggests a barrel twist-rate n for this bullet of either

$$n = 150 / 3.98 = 37.7 \text{ calibers (up to Mach 2.5).}$$

Substituting these values into our expression for e above:

$$e = (k_x^2 / n) * (CL_\alpha + CD_0)^2 / (CM_\alpha * CD_\alpha) = 0.002567$$

and

$$1/e = 389.5.$$

for achieving initial coning motion having “minimum coning angle” with this bullet right out of the muzzle of the rifle barrel at Mach 2.5.

[Curiously, $e \cdot k_x^{-2} = 0.02366$ for this old 168-grain Sierra International bullet, which is similar to the average long-range **ScF** value of **0.02322** for the Army 175.16-grain M118LR Special Ball and the Berger 175-grain Tactical 30-caliber bullets. Perhaps $e \cdot k_x^{-2}$ might also be similarly smaller for monolithic copper-alloy ULD bullets.]

Then, since $k_x^{-2} = 9.0 \text{ calibers}^{-2}$ for almost any modern monolithic rifle bullet, the maximum value of **n** for achieving initial hyper-stable flight is:

$$\begin{aligned} n &= (389.54/k_x^{-2}) \cdot (CL_\alpha + CD_0)^2 / (CM_\alpha \cdot CD_\alpha) \\ n &= 43.28 \cdot (CL_\alpha + CD_0)^2 / (CM_\alpha \cdot CD_\alpha). \end{aligned}$$

Coning Potential Energy

The orbital potential energy **P_C** of the coning rifle bullet can be formulated as

$$P_C = -\int F_C \cdot dr = k_C \cdot r^2 / 2 = q \cdot S \cdot \sin(\alpha) \cdot [CL_\alpha + CD] \cdot r / 2$$

or

$$P_C = q \cdot S \cdot D \cdot [CL_\alpha + CD] \cdot \alpha^2 / 2.$$

Because the harmonic coning motion is isotropic and the orbital motion of the CG of the bullet is circular (at least non-elliptical), the orbital kinetic energy **E_C** and potential energy **P_C** are ***always equal***:

$$E_C = (m/2) \cdot (D \cdot \omega_2)^2 \cdot \alpha^2 =$$

$$P_C = q \cdot S \cdot D \cdot [CL_\alpha + CD] \cdot \alpha^2 / 2$$

or

$$m \cdot D \cdot (\omega_2)^2 = q \cdot S \cdot [CL_\alpha + CD].$$

$$D = q \cdot S \cdot (CL_\alpha + CD_0) / [m \cdot (\omega_2)^2].$$

Recall from *Coning Theory* that

$$D = q \cdot S \cdot (CL_\alpha + CD_0) / [m \cdot (\omega_2)^2] \quad \text{QED.}$$

Evaluation of Slow-Mode Damping Factor

As shown above, the kinetic energy loss due to yaw-drag ΔE_α over a half coning cycle $T_2/2$ can be written as

$$\Delta E_\alpha(T_2/2) = q * S * V * \alpha^2 * C_{D_\alpha} * T_2/2.$$

We can also formulate the kinetic energy E_c of the orbital coning motion itself as

$$\begin{aligned} E_c &= (m/2) * (r * \omega_2)^2 = (m/2) * (D * \sin \alpha * \omega_2)^2 \\ &= (m/2) * (D * \omega_2)^2 * \alpha^2 \end{aligned}$$

where r is the coning radius of the CG of the bullet orbiting around a “mean CG” location moving along the “mean trajectory” of the bullet and D is the slowly varying coning distance of the CG of the bullet from its coning apex, each given in feet, so that $r = D * \sin(\alpha)$.

Now, as the coning angle α decreases (due to frictional damping) from its initial value α_0 to its final value α_1 at the completion of this half coning cycle, the change ΔE_c in orbital coning energy can be written as

$$\begin{aligned} \Delta E_c &= (m/2) * (D * \omega_2)^2 * (\alpha_0^2 - \alpha_1^2) \\ \Delta E_c &= m * (D * \omega_2)^2 * (\alpha_0 - \alpha_1) * (\alpha_0 + \alpha_1) / 2 \\ \Delta E_c &= m * (D * \omega_2)^2 * \alpha * \Delta \alpha \end{aligned}$$

where $\alpha_0 - \alpha_1 = \Delta \alpha > 0$, the *reduction* in coning angle due to damping, and $(\alpha_0 + \alpha_1)/2 = \alpha$, the average coning angle over this half cycle.

We now hypothesize that, at least in **hyper-stable flight** in which no nutation needs damping, and for **dynamically stable bullets**, the average loss in “forward motion” kinetic energy ΔE_α over any half coning cycle due to flying with an aerodynamic angle-of-attack α causes, and the average “frictional damping” decrease in coning energy ΔE_c during that same half coning cycle, must be proportional to each other. That is to say, we are tentatively assuming that a small portion e of the yaw-drag of the bullet

directly causes the damping of its coning angle α in **steady-state, minimum coning angle, hyper-stable flight**.

If this hypothesis is true, we can set $\Delta E_c = e \cdot \Delta E_\alpha$ over any particular half coning cycle, where the constant fraction e is greater than **zero** but not greater than **1.0**, and so that

$$m \cdot (D \cdot \omega_2)^2 \cdot \alpha \cdot \Delta \alpha = q \cdot S \cdot V \cdot \alpha^2 \cdot e \cdot C_{D_\alpha} \cdot T_2/2$$

or, dividing through by α^2 and by $[m \cdot (D \cdot \omega_2)^2]$,

$$(\Delta \alpha)/\alpha = (T_2/2) \cdot [q \cdot S \cdot V \cdot e \cdot C_{D_\alpha}] / [m \cdot (D \cdot \omega_2)^2].$$

We recognize that this expression has the form of the classic exponential damping of the coning angle α which was discussed above:

$$\alpha(t) = \alpha(0) \cdot \exp[-\lambda_2 \cdot t]$$

with $\lambda_2 = [q \cdot S \cdot V \cdot e \cdot C_{D_\alpha}] / [m \cdot (D \cdot \omega_2)^2]$.

If we replace the half coning period $T_2/2$ with a small increment in time dt , and replace $\Delta \alpha$ per half coning cycle with a small decrement $-d\alpha$ in α , then in the limit as dt approaches zero, this expression becomes

$$d\alpha/\alpha = -\lambda_2 \cdot dt$$

After integrating both sides from **0** to t ,

$$\ln[\alpha(t)/\alpha(0)] = -\lambda_2 \cdot t$$

Or, after exponentiating

$$\alpha(t) = \alpha(0) \cdot \exp[-\lambda_2 \cdot t] \quad [\text{QED}].$$

Thus, we have derived the long-accepted damping relationship from the basic physics of our hypothesis that a portion of the yaw-drag causes the damping of the coning angle for dynamically stable bullets in hyper-stable flight.

If only a fraction e ($0 < e \leq 1.0$) of this extra yaw-drag induced kinetic energy loss is actually responsible for frictional damping of the coning

angle $\alpha(t)$, we accommodate that simply by using e^*CD_α in the above expression for λ_2 :

$$\lambda_2 = [q^*S^*V^*e^*CD_\alpha]/[m^*(D^*\omega_2)^2].$$

Two basic magnitude relations from Coning Theory allow simplification of this expression above for the slow-mode damping factor λ_2 . We know that the coning distance D (in feet) is given by

$$D = q^*S^*(CL_\alpha + CD_0)/[m^*(\omega_2)^2]$$

and, we know that the magnitude of the coning rate ω_2 is given by

$$\omega_2 = q^*S^*d^*CM_\alpha/(I_x^*\omega)$$

where, from Tri-Cyclic Theory, the expression for angular momentum of the spinning bullet can be written as

$$I_x^*\omega = I_y^*(\omega_2 + \omega_1) = I_y^*\omega_2^*(R + 1).$$

Substituting in the denominator of the expression for ω_2 :

$$\omega_2 = q^*S^*d^*CM_\alpha/[I_y^*\omega_2^*(R + 1)]$$

or, multiplying by ω_2

$$(\omega_2)^2 = q^*S^*d^*CM_\alpha/[I_y^*(R + 1)].$$

Now, we can reformulate the coning distance D as

$$D = I_y^*(R + 1)^*q^*S^*(CL_\alpha + CD_0)/[m^*q^*S^*d^*CM_\alpha]$$

$$D = I_y^*(R + 1)^*(CL_\alpha + CD_0)/(m^*d^*CM_\alpha).$$

And, $(D^*\omega_2)^2$ can be expressed as

$$(D^*\omega_2)^2 = \{[I_y^*(R + 1)^*(CL_\alpha + CD_0)]^2 * q^*S^*d^*CM_\alpha\} / \{(m^*d^*CM_\alpha)^2 * I_y^*(R + 1)\}$$

$$(D^*\omega_2)^2 = \{q^*S^*I_y^*(R + 1)^*(CL_\alpha + CD_0)^2\}/\{m^2 * d^*CM_\alpha\}$$

Substituting for $(D*\omega_2)^2$ in our expression for λ_2 , we have

$$\lambda_2 = [q*S*V*e*CD_\alpha]/[m*(D*\omega_2)^2]$$

$$\lambda_2 = \{(m^2 * d * CM_\alpha) * [q * S * V * e * CD_\alpha]\} / \{m * q * S * I_y * (R + 1) * (CL_\alpha + CD_0)^2\}$$

Collecting terms

$$\lambda_2 = \{m * d * e * V / [I_y * (R + 1)]\} * \{CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2\}$$

Let a Stability Coefficient CS_α stand for the combined aeroballistic coefficients expression for any particular Mach speed:

$$CS_\alpha = CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2$$

Then

$$\lambda_2 = \{m * d * e * V / [I_y * (R + 1)]\} * CS_\alpha$$

From Tri-Cyclic Theory

$$(I_x/I_y) * \omega = \omega_1 + \omega_2 = \omega_2 * (R + 1)$$

$$\text{so, } R + 1 = (I_x/I_y) * \omega / \omega_2 = (I_x/I_y) * f / f_2$$

$$\text{and } I_y * (R + 1) = I_x * f / f_2.$$

So, the expression for λ_2 can now be written as

$$\lambda_2 = \{m * d * e * V / [I_x * f]\} * f_2 * CS_\alpha$$

$$\text{But, } I_x = m * d^2 * k_x^2$$

so, the expression for λ_2 can be rewritten as

$$\lambda_2 = \{V / [d * f]\} * e * k_x^{-2} * f_2 * CS_\alpha$$

Right out of the muzzle

$$f = V_0 / (n * d) \text{ revolutions/second}$$

$$\text{or } n = V_0 / (d * f) \text{ calibers/turn.}$$

So, using **initial values** for each flight variable,

$$\lambda_2 = n * e * k_x^{-2} * f_2 * CS_\alpha .$$

For the well studied 30-caliber 168-grain Sierra International bullet, for example, at an initial airspeed of Mach 2.5:

$$k_x^{-2} = 9.218 \text{ calibers}^{-2}$$

$$I_y/I_x = 7.441$$

$$CL_\alpha = 2.850$$

$$CD_0 = 0.320$$

$$CM_\alpha = 2.560$$

$$CD_\alpha = 4.400$$

$$n = 38.96 \text{ calibers/turn (or 12 inches/turn)}$$

$$f_1 + f_2 = 2800/7.441 = 376.3 \text{ hz}$$

$$S_g = 1.75$$

$$R = 4.79$$

$$f_2 = (f_1 + f_2)/(R + 1) = 65.0 \text{ hz}$$

And, at Mach 2.5

$$CS_\alpha = CM_\alpha * CD_\alpha / (CL_\alpha + CD_0)^2 = 1.121$$

$$e = (k_x^2/n) * CS_\alpha = 0.002567.$$

From data published by Robert L. McCoy of the Ballistics Research Lab (BRL) at Aberdeen Proving Ground in Maryland, the pertinent aeroballistics coefficients for this old 168-grain bullet as a function of airspeed in Mach Numbers were as shown in the table below.

<u>30-caliber 168-grain Sierra International (per McCoy)</u>							
					(spin damp)		
	<u>Mach No.</u>	<u>CMa</u>	<u>CDa</u>	<u>CLa</u>	<u>CD0</u>	<u>Clp</u>	<u>CSa</u>
	2.50	2.56	4.40	2.85	0.320	-0.0068	1.1209
	2.20	2.69	5.40	2.68	0.339	-0.0073	1.5937
	2.00	2.79	6.10	2.58	0.350	-0.0075	1.9824
	1.80	2.88	6.80	2.45	0.365	-0.0080	2.4714
	1.60	2.98	7.30	2.32	0.385	-0.0083	2.9731
	1.40	3.06	7.60	2.15	0.410	-0.0088	3.5486
	1.20	3.12	6.50	1.90	0.434	-0.0095	3.7228
	1.10	3.15	3.60	1.70	0.447	-0.0098	2.4601
	1.05	3.17	3.10	1.55	0.449	-0.0099	2.4592
	1.00	3.24	3.00	1.35	0.430	-0.0100	3.0678
	0.95	3.45	2.90	1.30	0.240	-0.0103	4.2187
	0.90	3.43	2.90	1.35	0.160	-0.0105	4.3625
	0.85	3.40	2.90	1.40	0.142	-0.0107	4.1468
	0.80	3.38	2.90	1.45	0.140	-0.0108	3.8772
	0.50	3.26	2.90	1.63	0.140	-0.0125	3.0177
	0.00	3.05	2.90	1.75	0.140	-0.0150	2.4761

So, based on these aeroballistic parameters,

$$\lambda_2 = n * e * k_x^{-2} * f_2 * CS_\alpha$$

$$\lambda_2 = 59.9 \text{ seconds}^{-1}$$

and

$$\lambda_s = -\lambda_2 * d / V_0 = -0.000549 \text{ calibers}^{-1}.$$

For critical damping during each full coning cycle we would need a damping factor of

$$[\lambda_2]_{\text{crit}} = f_2 = 65.0 \text{ sec}^{-1}$$

So, this λ_2 damping would be a bit less than **critical damping** of the coning angle α . This sub-critical damping of the coning angle requires only about **0.2567-percent** of the energy loss due to yaw-drag.

We can calculate a barrel twist rate n (calibers/turn) for just critical damping as:

$$\begin{aligned} [n]_{\text{crit}} &= [\lambda_2]_{\text{crit}}/[e \cdot k_x^{-2} \cdot f_2 \cdot CS_\alpha] \\ [n]_{\text{crit}} &= 1/[e \cdot k_x^{-2} \cdot CS_\alpha] \\ &= 37.7 \text{ calibers/turn (per Greenhill)} \end{aligned}$$

or $[n]_{\text{crit}} = 37.7 \cdot (0.308 \text{ in/cal}) = 11.6 \text{ inches/turn.}$

Unfortunately, the old 30-caliber, 168-grain Sierra International bullet was **not** actually dynamically stable at Mach 2.5 airspeed due to several bullet design errors. These calculations are shown as if it were stable simply because it is one of the few bullets for which we have the complete set aeroballistic coefficient data. The above formulation for λ_2 does not apply for dynamically unstable bullets. The slow-mode damping factor for that particular bullet at Mach 2.5 was actually negative (in the formulation used here).

For initial critical damping of the coning motion of any rifle bullets which are dynamically stable, we can formulate the barrel twist-rate required $[n]_{\text{crit}}$ in **calibers/turn**. Since $[\lambda_2]_{\text{crit}} = f_2$, the expression for $[n]_{\text{crit}}$ reduces to:

$$[n]_{\text{crit}} = 1/(e \cdot k_x^{-2} \cdot CS_\alpha)$$

with $CS_\alpha = CM_\alpha \cdot CD_\alpha / (CL_\alpha + CD_0)^2$

and all coefficients evaluated at muzzle speed.

With the constant fraction $e = 0.0023345$ and $k_x^{-2} \approx 9.0 \text{ calibers}^{-2}$ for monolithic VLD and ULD rifle bullets, the expression for $[n]_{\text{crit}}$ becomes

$$[n]_{\text{crit}} = 47.6 \text{ calibers}/CS_\alpha.$$

As shown in the table above for the old Sierra International bullet, the Damping Coefficient CS_α varied for different muzzle speeds from **1.121** at Mach 2.5 up to **3.723** at Mach 1.20.

For long-nosed monolithic copper-alloy ULD bullets, we can expect CD_α to be larger and $(CL_\alpha + CD_0)$ to be smaller at high Mach-speeds out of the

muzzle, so expecting a value of about **2.0** to **2.5** for **CS_α** is not unreasonable for these modern bullets.

As the Damping Coefficient **CS_α** increases, the barrel twist-rate required for critical damping of the coning angle **α** , and thus for achieving early hyper-stable bullet flight, must get “faster.” That is, **$[n]_{crit}$** in **calibers/turn** must get smaller.

For those firing monolithic copper-alloy bullets at Mach 3.0 to Mach 3.5, the single best recommended barrel twist-rate should be:

$$[n]_{crit} \approx 20 \text{ calibers/turn.}$$

By ensuring critical damping of the coning angle initially, a bullet fired from a barrel having **19 to 21 calibers per turn** twist-rate and entering the undisturbed ambient atmosphere a few yards ahead of the rifle with **zero** yaw attitude and **zero** yaw-rate should achieve hyper-stability initially and maintain it throughout its flight to a long-range target. This copper-alloy ULD bullet would be flying with minimum aerodynamic drag due only to its designer-minimized zero-yaw coefficient of drag **CD_0** all the way to its maximum-range target.

The initial gyroscopic stability **S_g** of such a monolithic ULD bullet fired from a barrel having this critical twist-rate should be between **2.5 (R = 8.9)** and **3.0 (R = 10)**. The initial dynamic stability for these bullets should then be between **0.36 (R = 8.9)** and **0.33 (R = 10)**. These bullets should then be exceedingly stable in transiting the turbulent transonic speed region far downrange and should then continue flying with minimum yaw (coning angle) as reasonably good subsonic bullets.